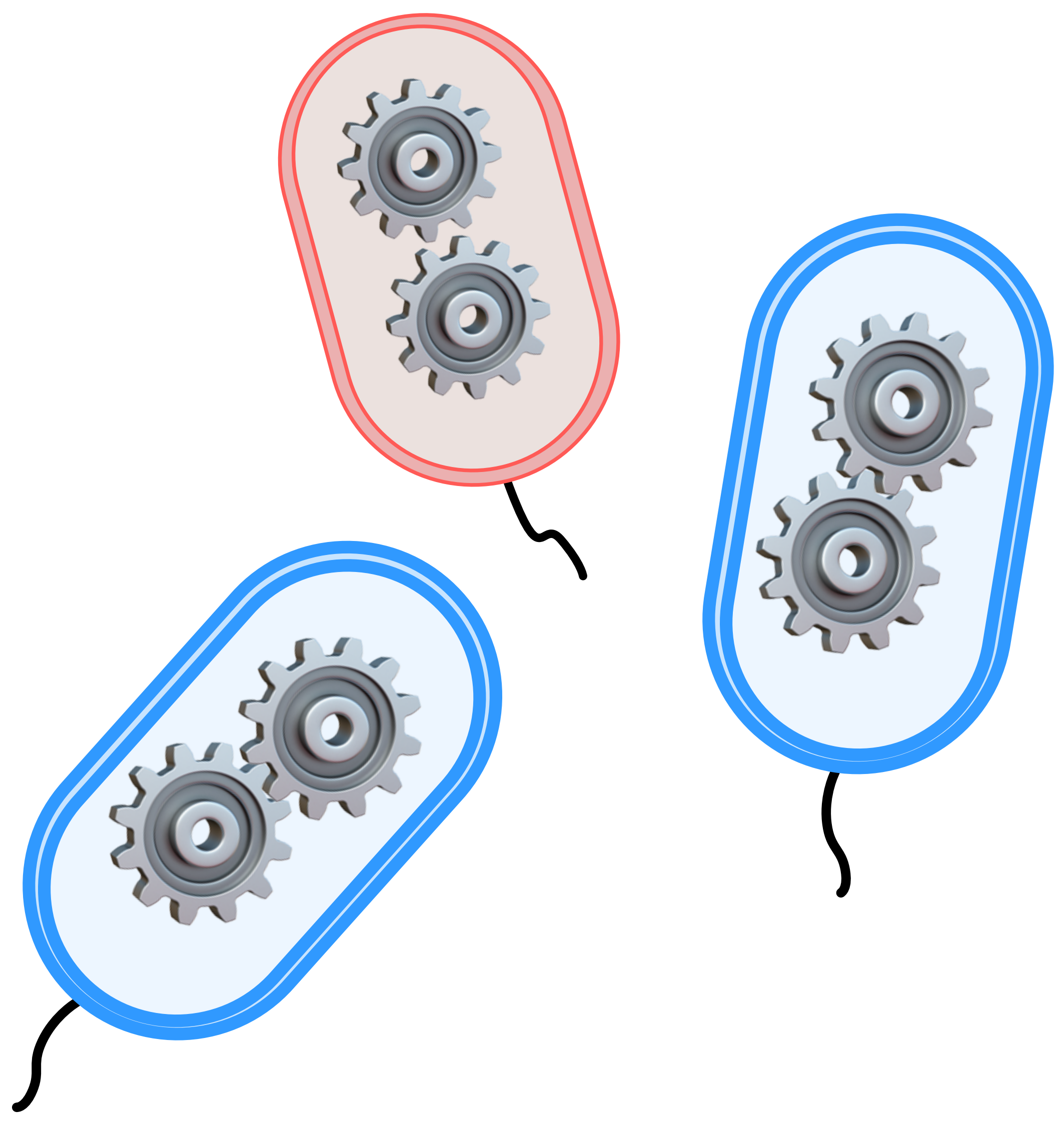
Majority consensus in stochastic populations

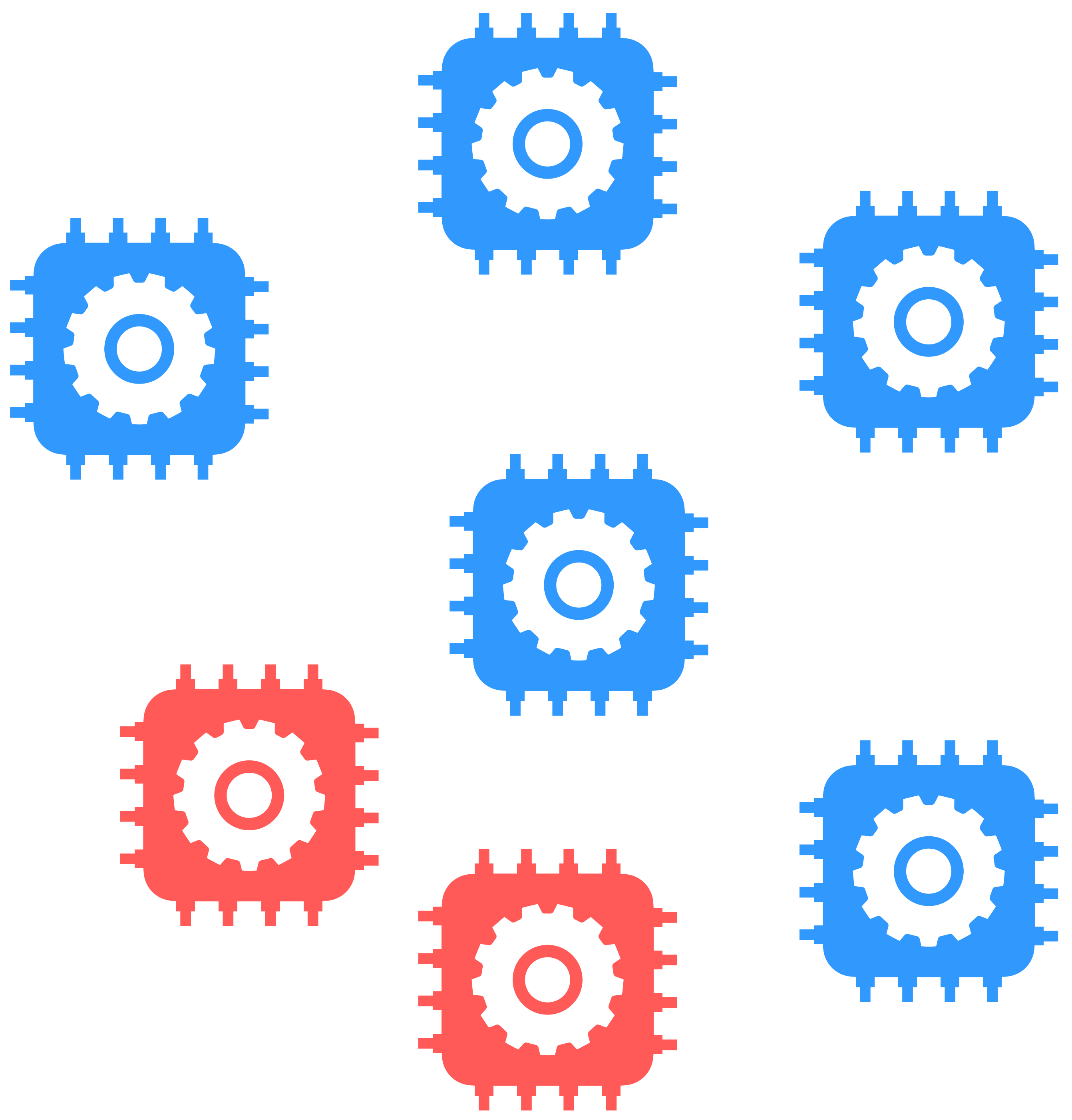


Joel Rybicki
Humboldt University of Berlin

Based on joint work with:
Victoria Andaur · Janna Burman · Matthias Függer
Manish Kushwaha · Bilal Manssouri · Thomas Nowak

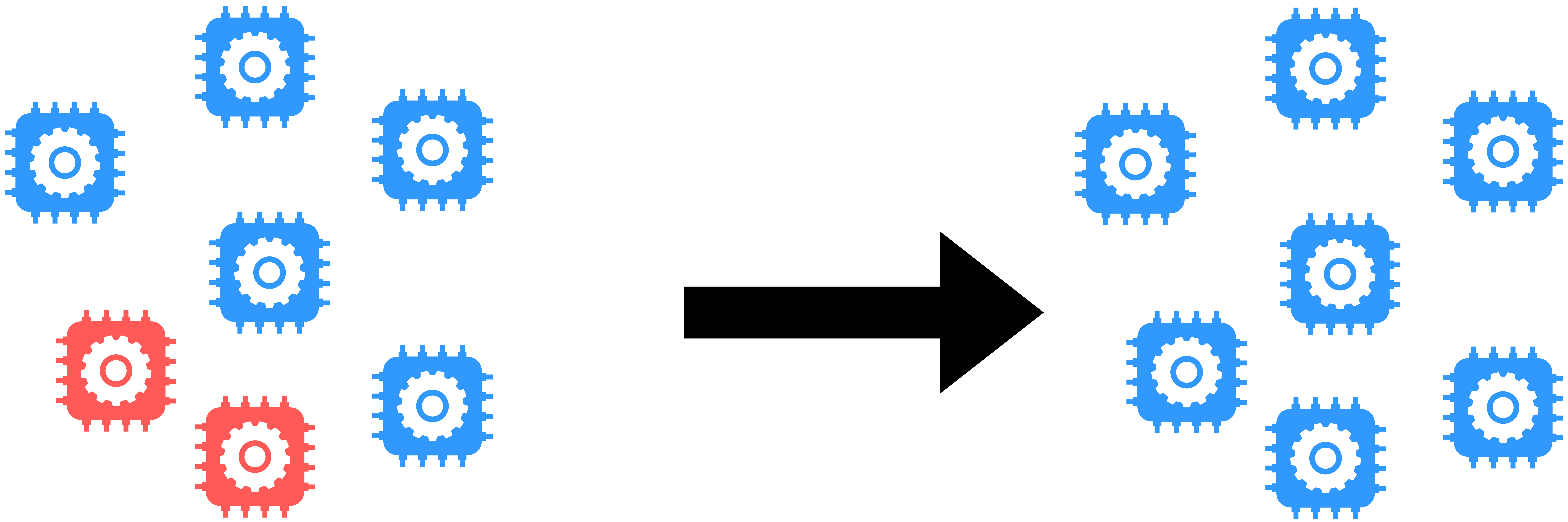
WAND 2024
November 1, 2024

Majority consensus



Task: output the *initial* majority value

Majority consensus



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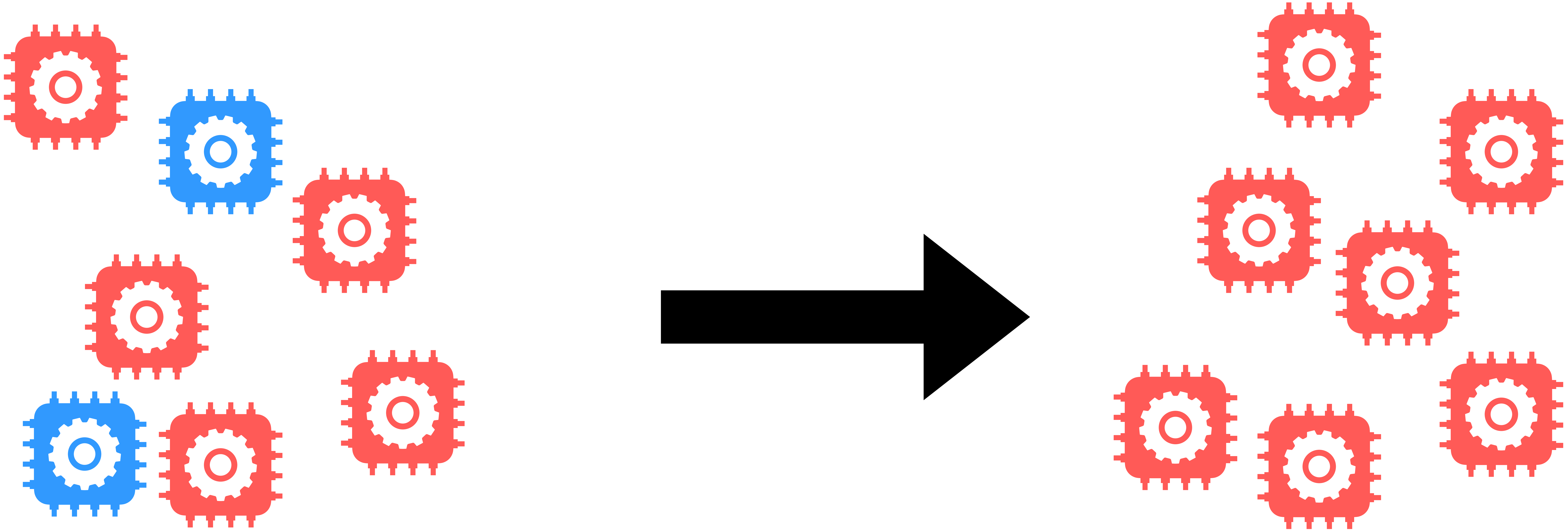
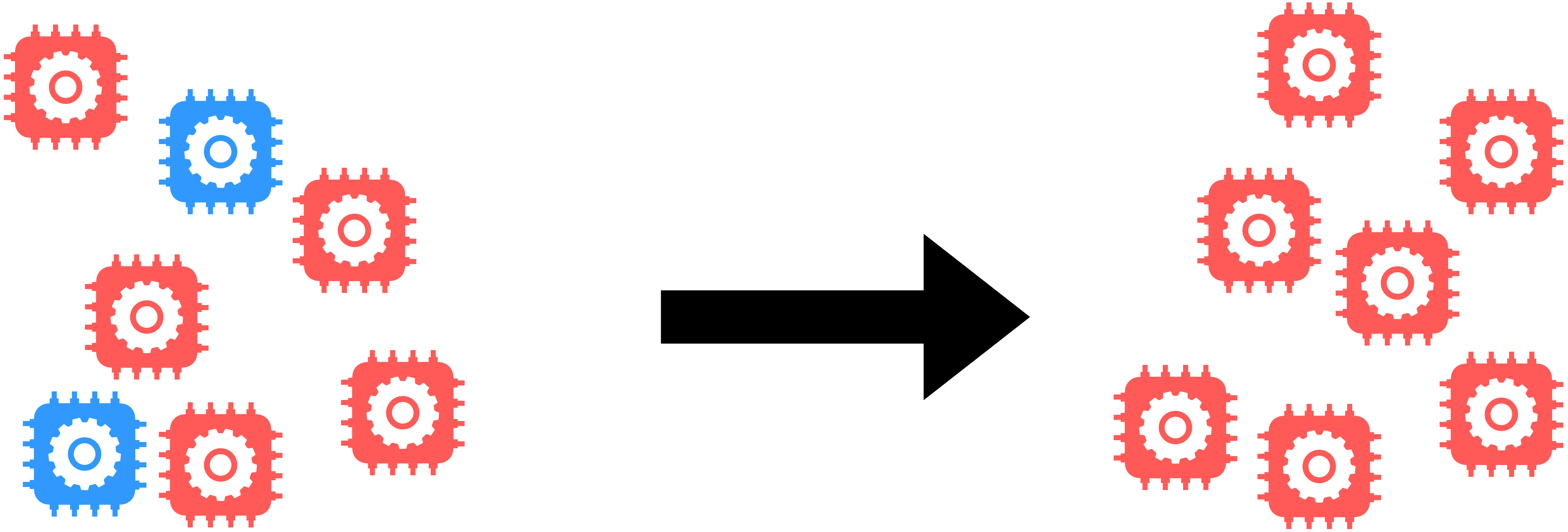
Majority consensus



Task: output the *initial* majority value

Majority consensus



Question: Given $\Delta = |A_0 - B_0|$,
how efficiently/likely can we reach majority consensus?

Scanning electron micrograph of *Escherichia coli*. Credit: NIAID

Credit: M. Függer

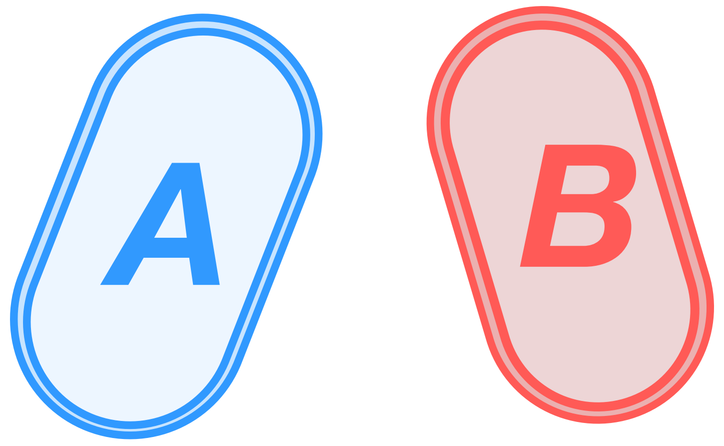
Distributed algorithm

=

engineered microbial community

Microbial majority consensus

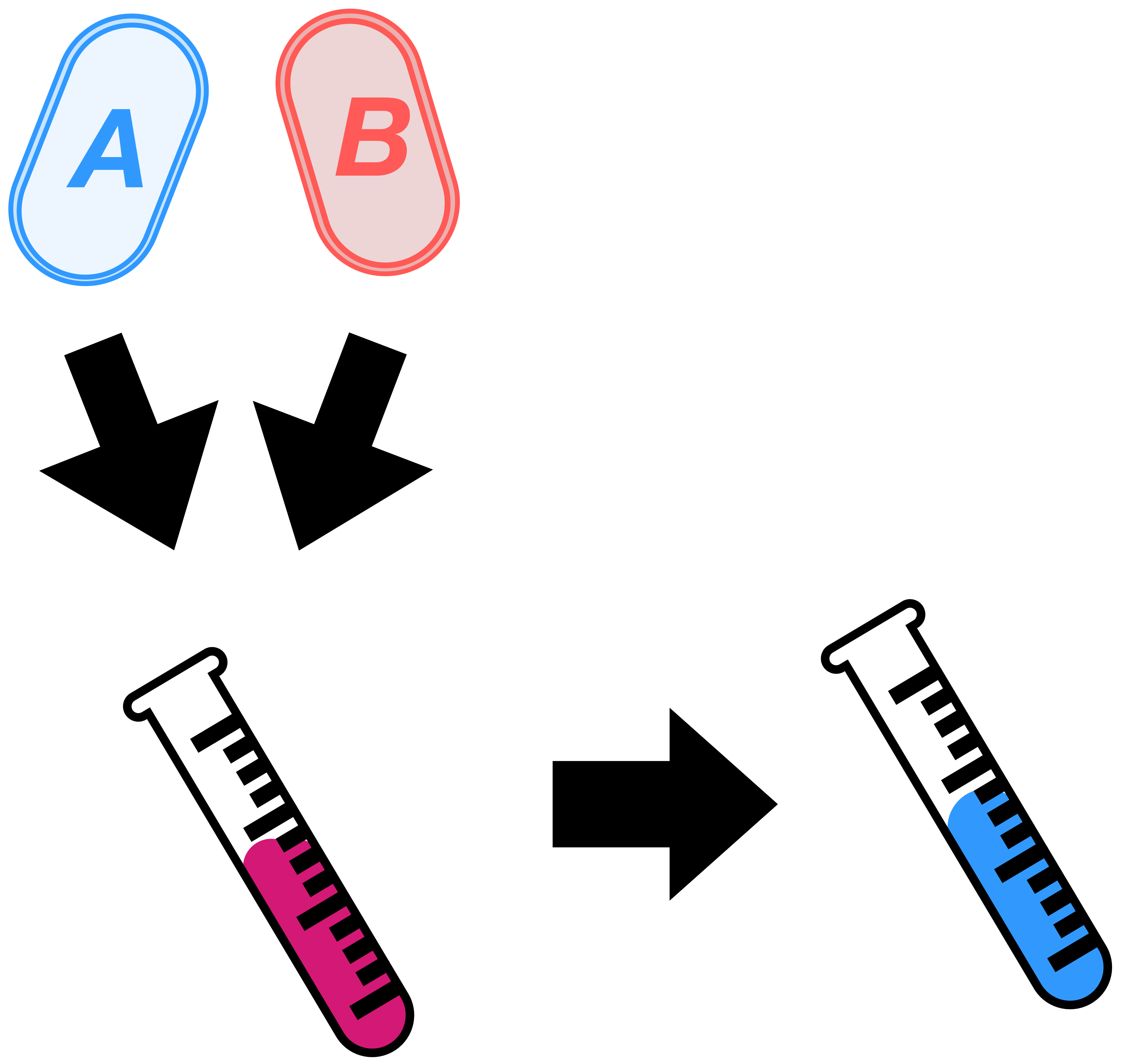
- **Inputs and outputs:**
  Two distinct microbial species



Microbial majority consensus

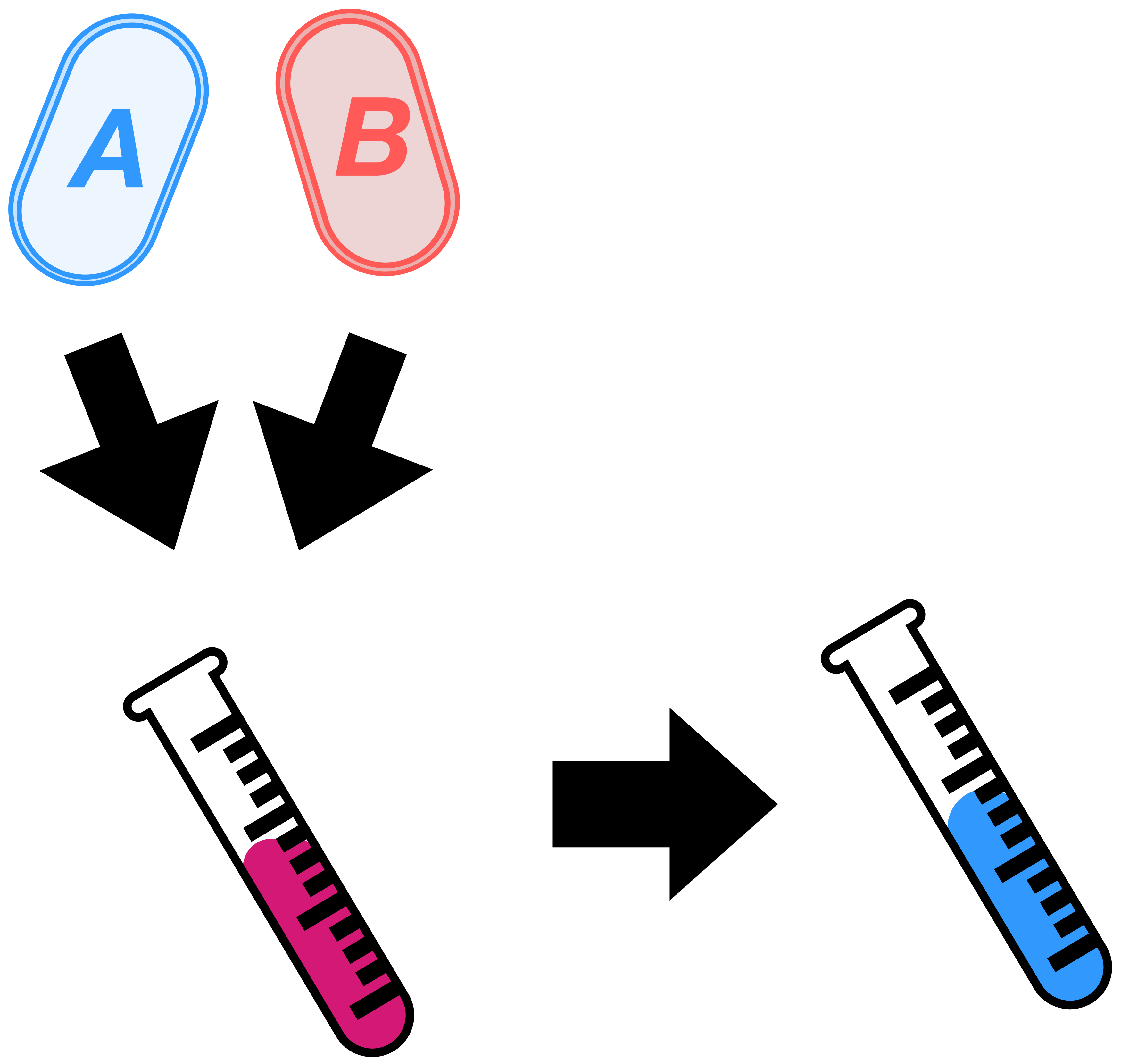
- **Inputs and outputs:**
  Two distinct microbial species

- **Well-mixed system (CRN)**
  stochastic interactions



Microbial majority consensus

- **Inputs and outputs:**
  Two distinct microbial species
- **Well-mixed system (CRN)**
  stochastic interactions
- **Microbial species:**
  biological population dynamics!



Majority consensus in distributed computing

- **Approximate majority** e.g.
  — Angluin, Aspnes and Eisenstat (DISC 2007)
  — Condon, Hajiaghayi, Kirkpatrick and Maňuch (Natural Computing 2020)

- **Exact majority** e.g.
  — Draief and Vojnović (INFOCOM 2012)
  — Alistarh and Gelashvili (ICALP 2015)
  — Doty, Eftekhari, Gąsieniec, Severson, Uznański, and Stachowiak (FOCS 2021)

- **Plurality consensus** e.g.
  — Becchetti, Clementi, Natale, Pasquale & Silvestri (SODA 2014)
  — Bankhamer, Berenbrink, Biermeier, Elsässer, Hosseinpour, Kaaser & Kling (SODA 2022)

Majority consensus in synthetic biology

- Majority consensus ≈ **state detection/signal amplification**

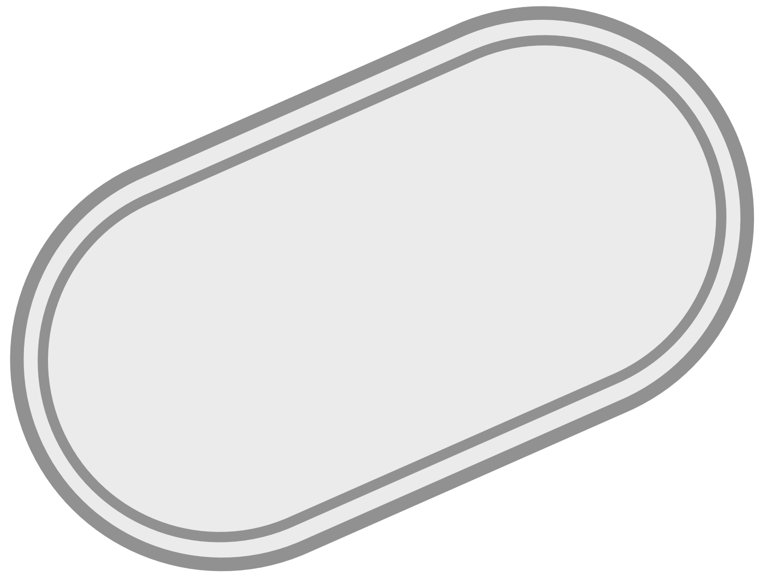
  Alnahhas et al., Nature Communications (2020), Cho et al. DISC (2019)

- Genetic modules exist to program chosen **ecological interactions**

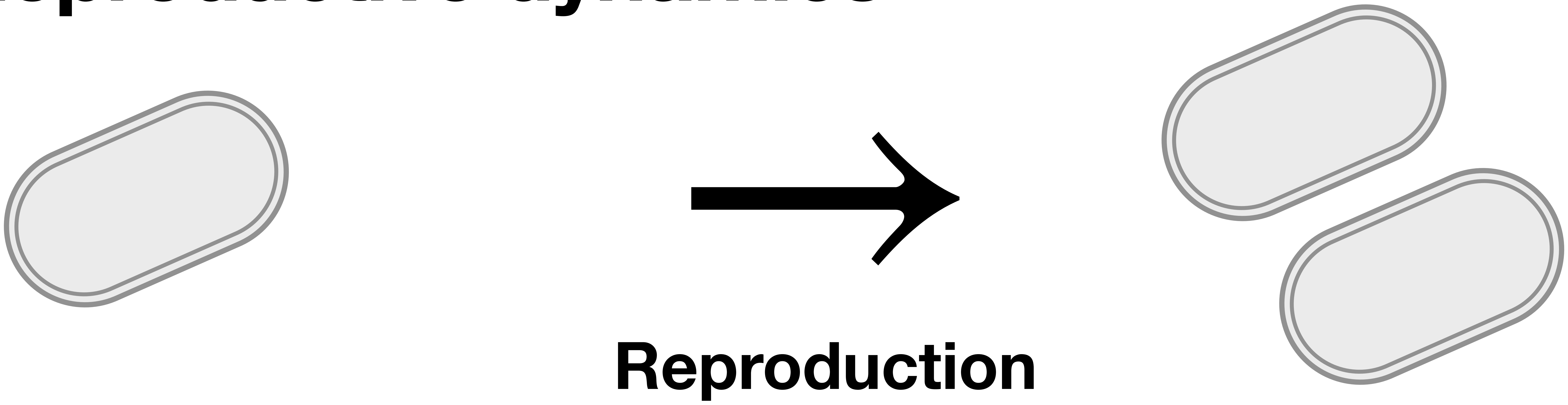
  Li et al., Methods in Ecology and Evolution (2023)

Are biological cells different from digital computers?

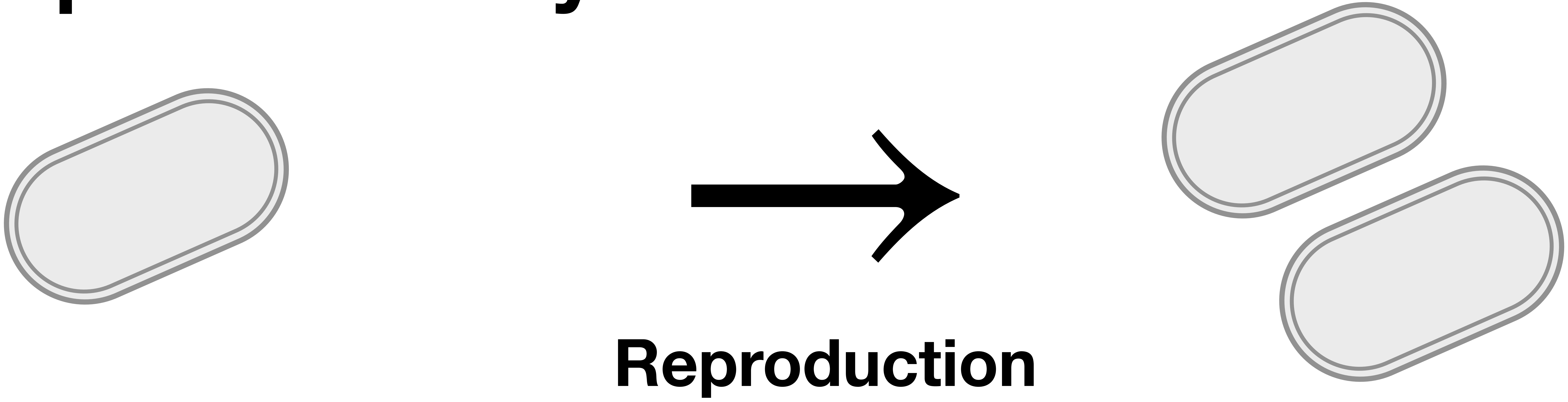
Reproductive dynamics

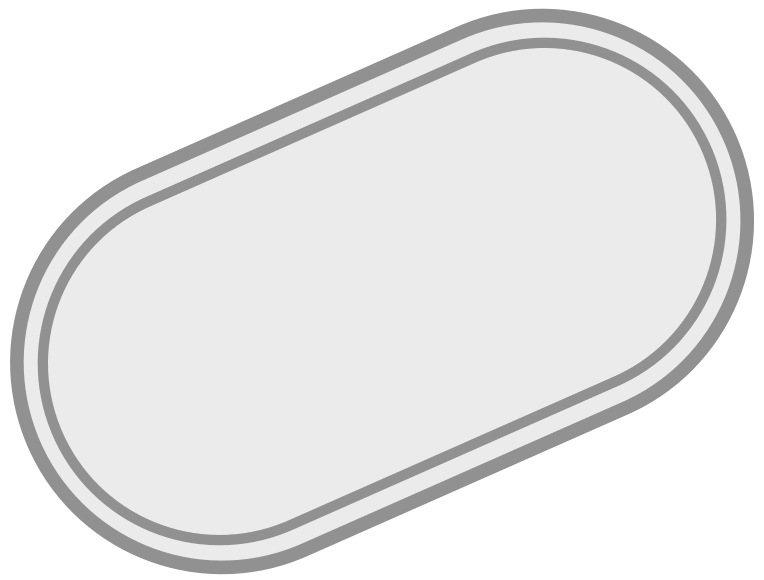


Reproductive dynamics

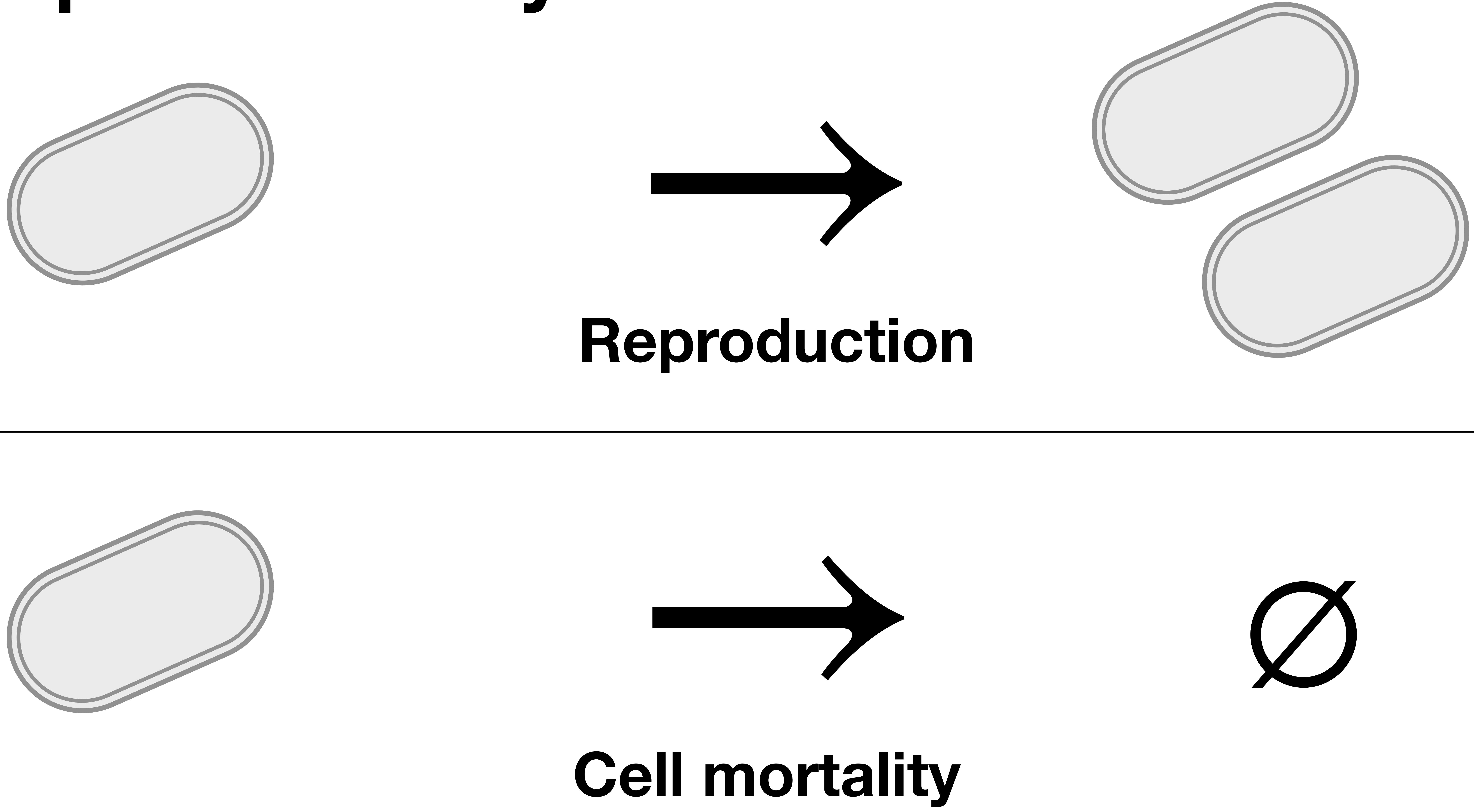


Reproductive dynamics

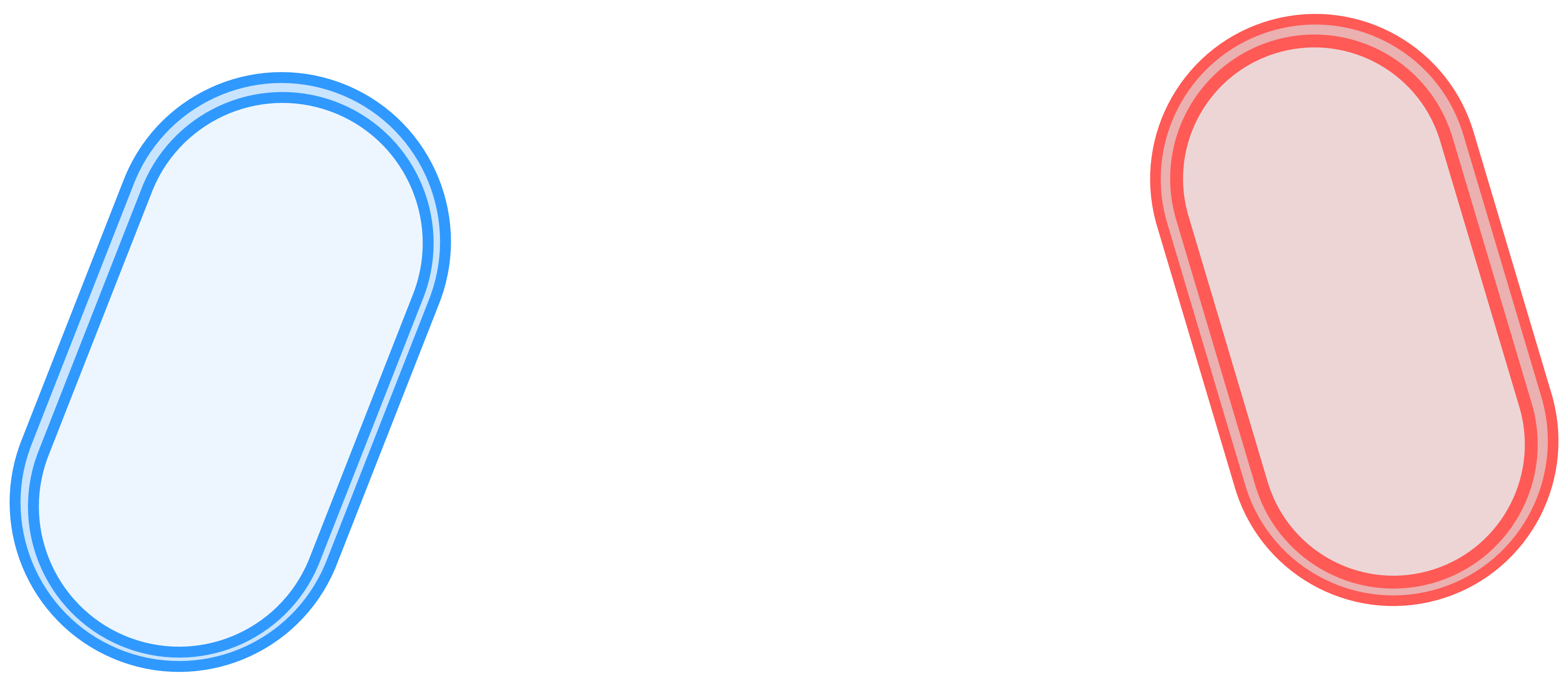




Reproductive dynamics



Competition



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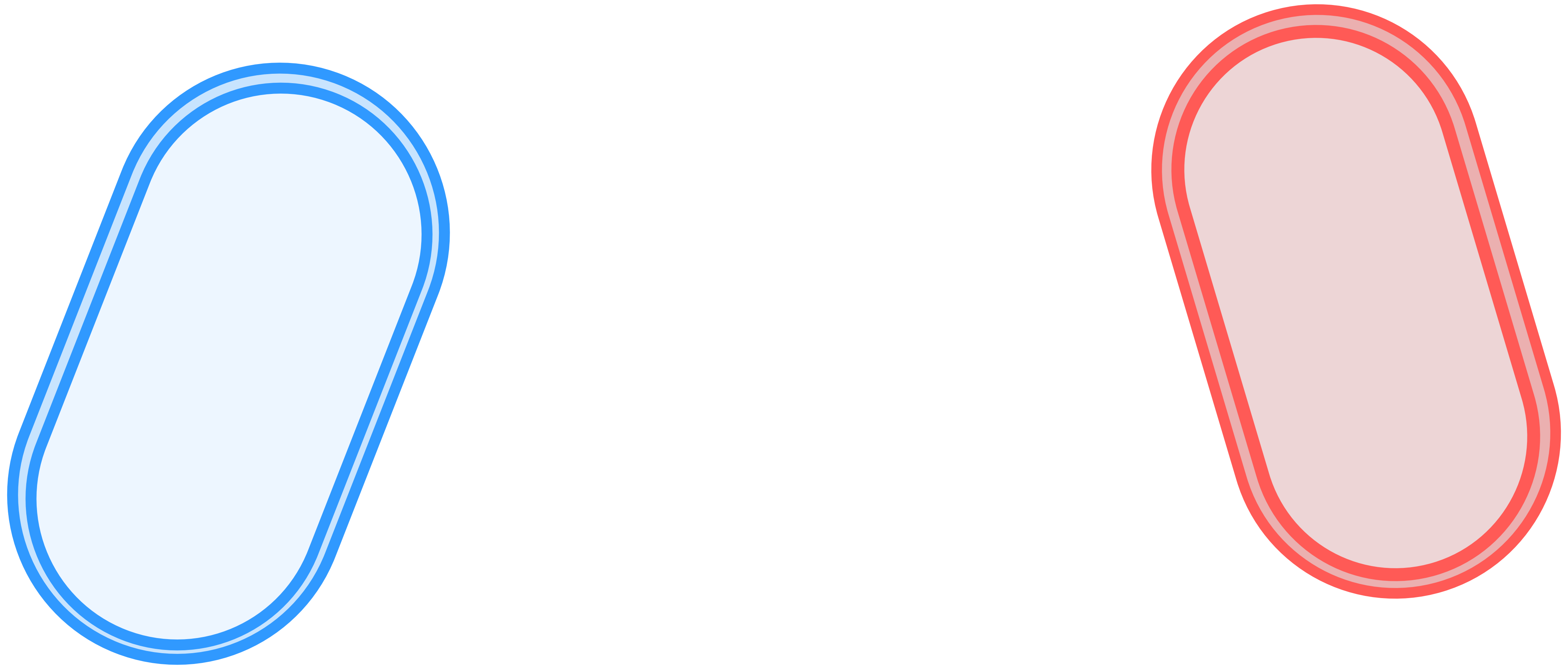
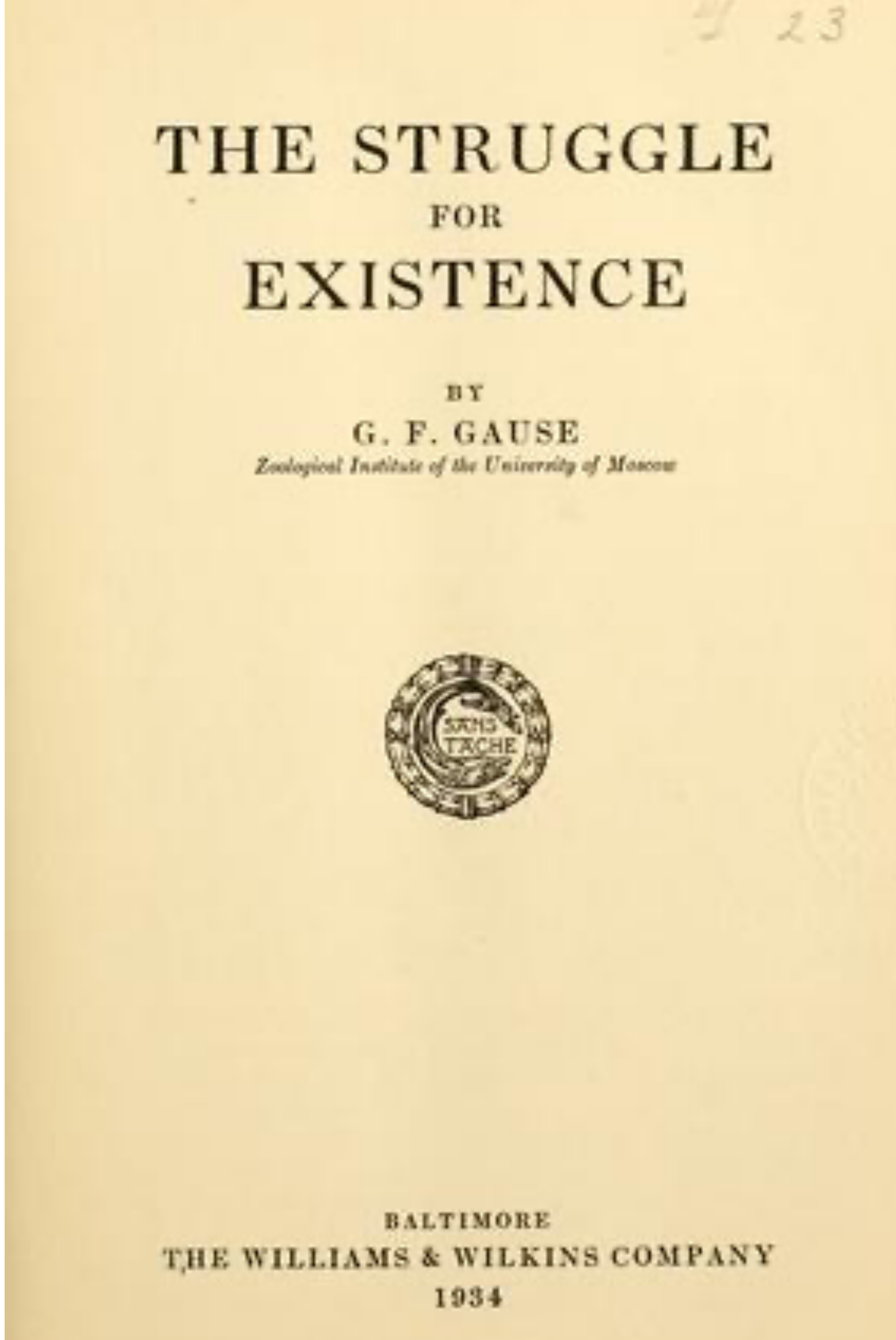




THE STRUGGLE
FOR
EXISTENCE

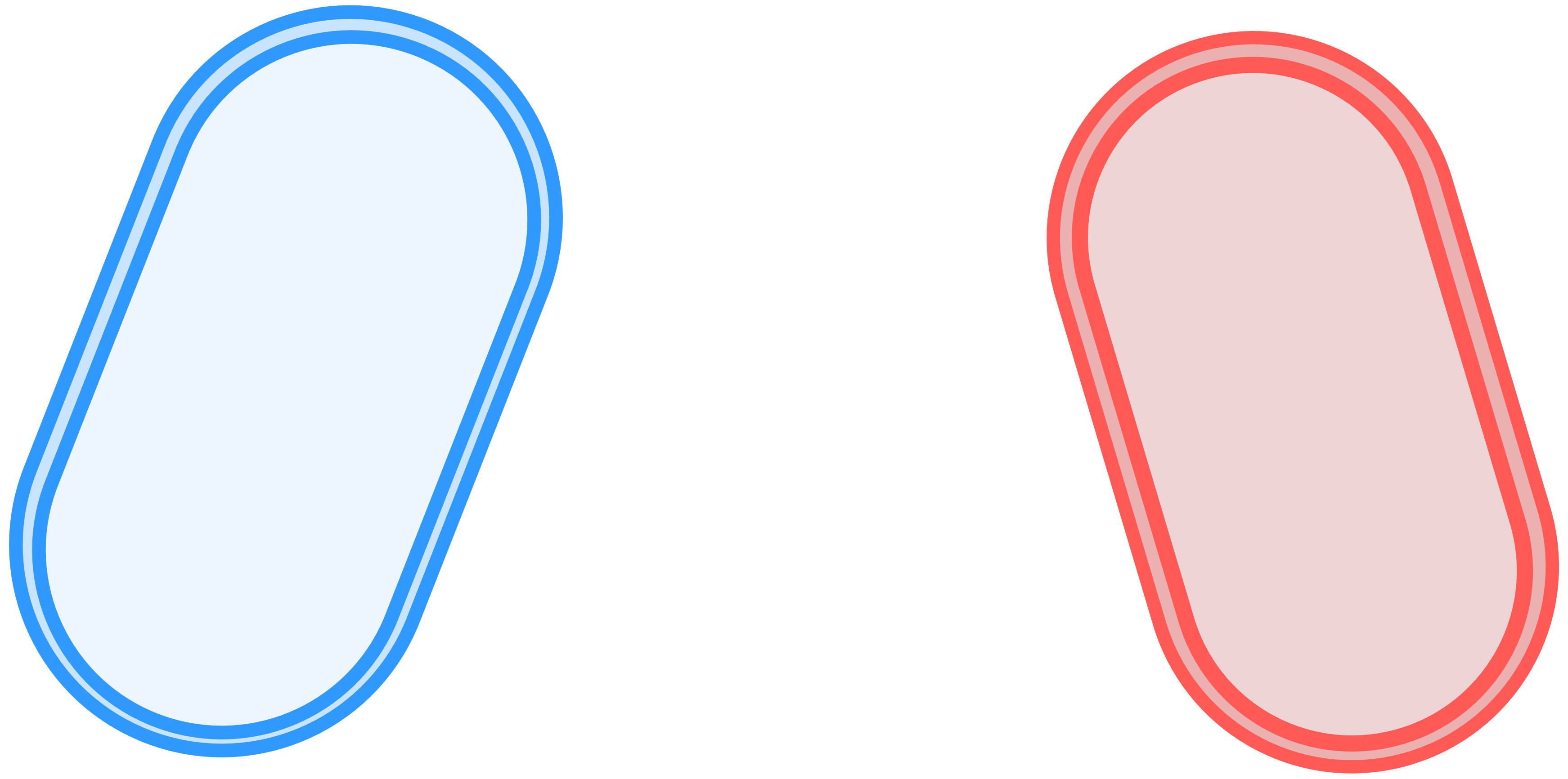
BY
G. F. GAUSE
Zoological Institute of the University of Moscow

BALTIMORE
THE WILLIAMS & WILKINS COMPANY
1934

Competition

- **exploitative competition:** competition for common resources (nutrients, space, …)

Competition



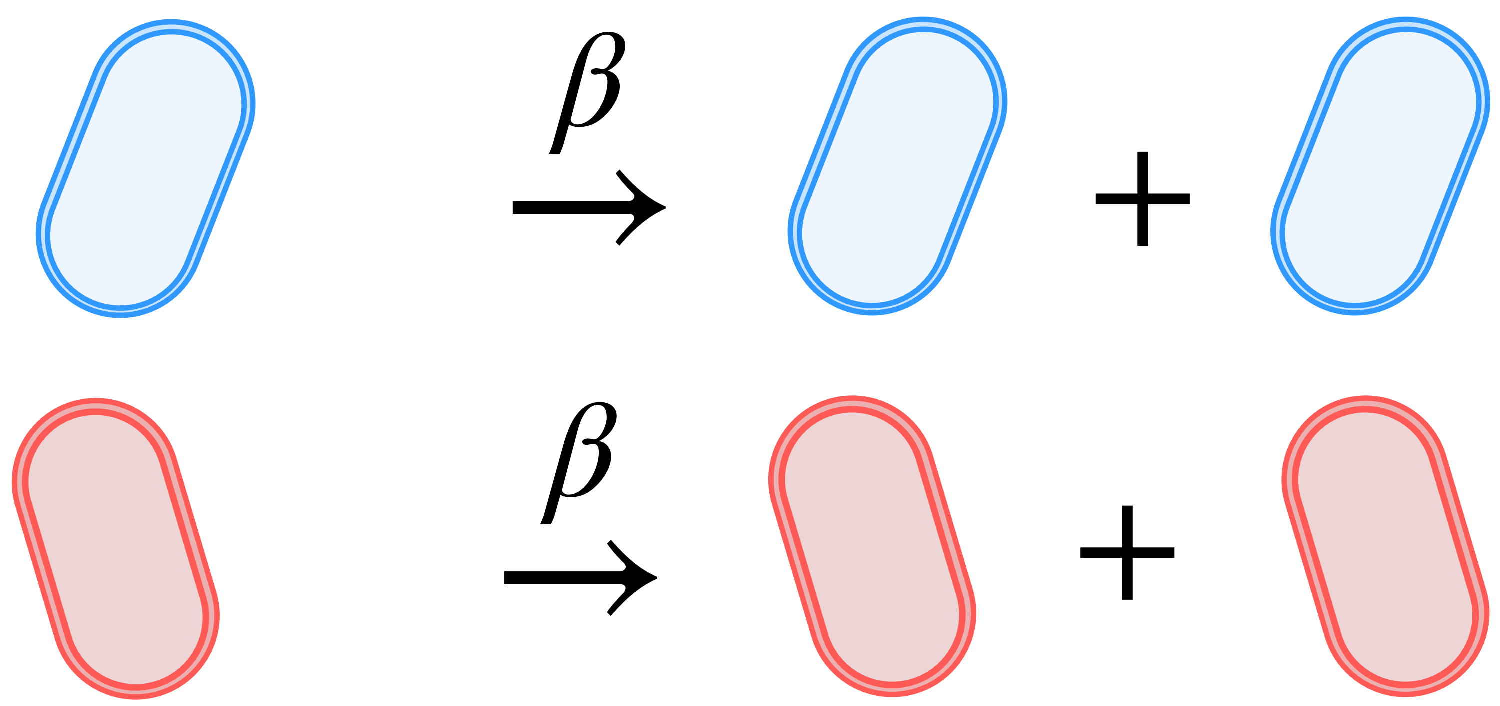
- **interference competition:** actively *interfere* with others' attempts to utilise resources

How does **demographic noise** and **competition** impact the performance of majority consensus dynamics?

Stochastic, competitive Lotka—Volterra models

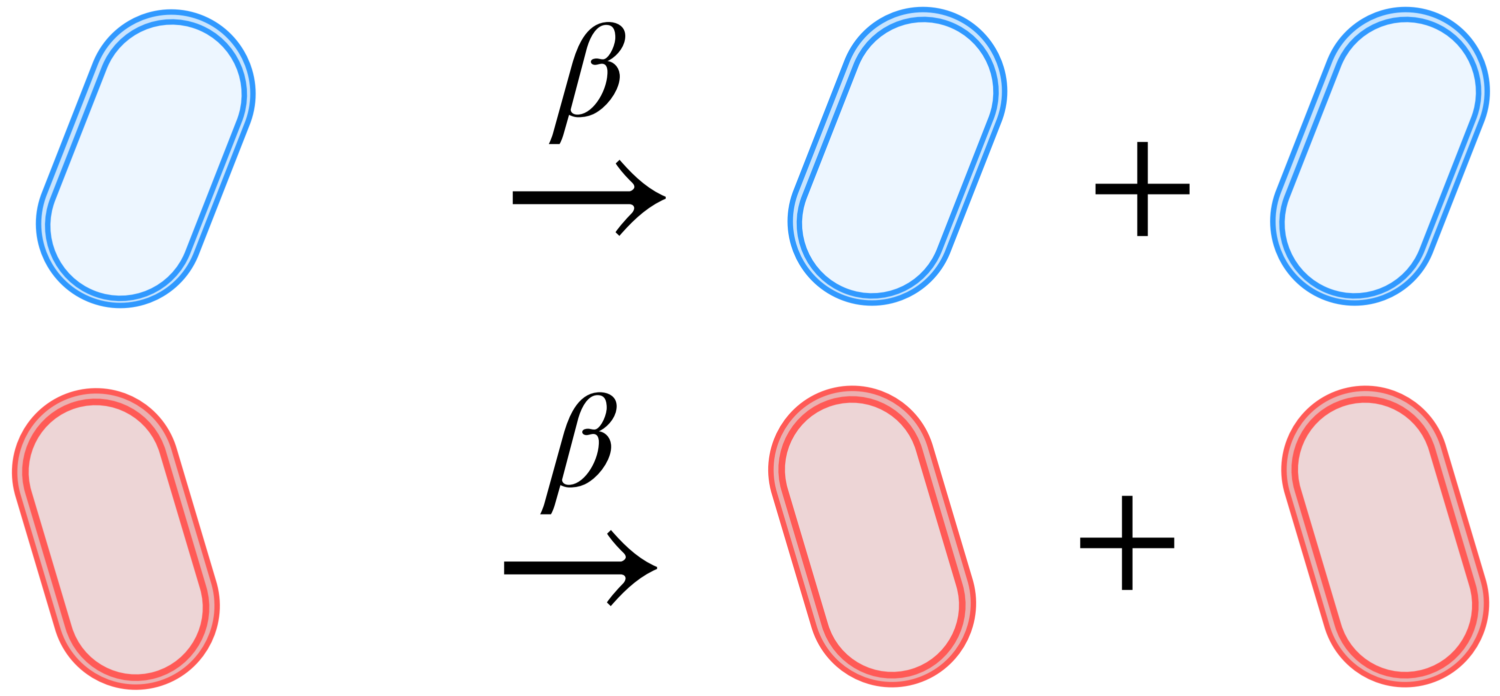
Competitive LV dynamics

Reproduction

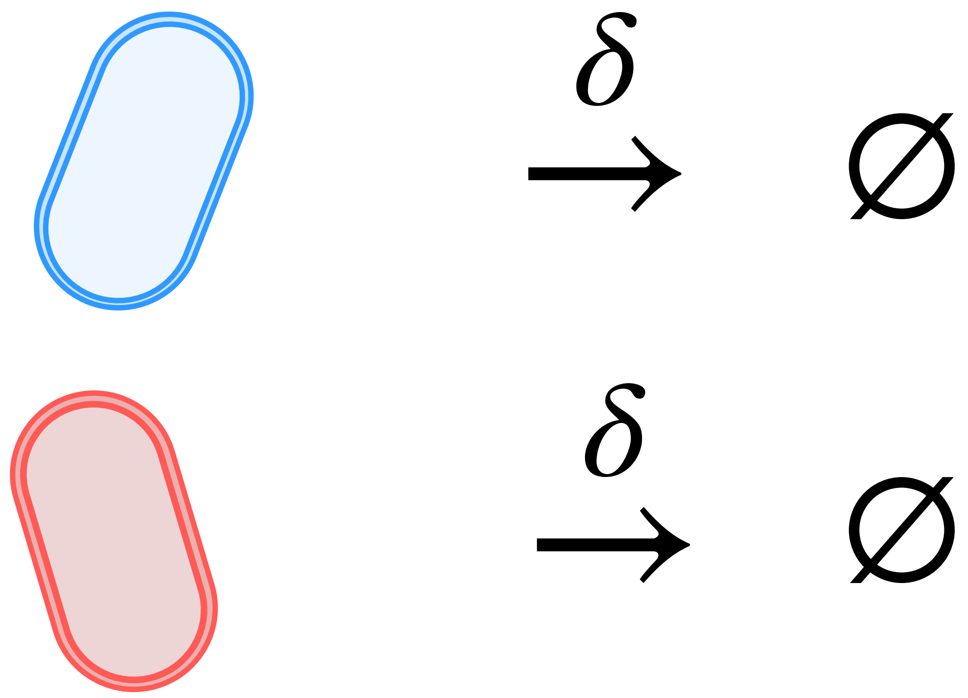


Competitive LV dynamics

Reproduction



Mortality



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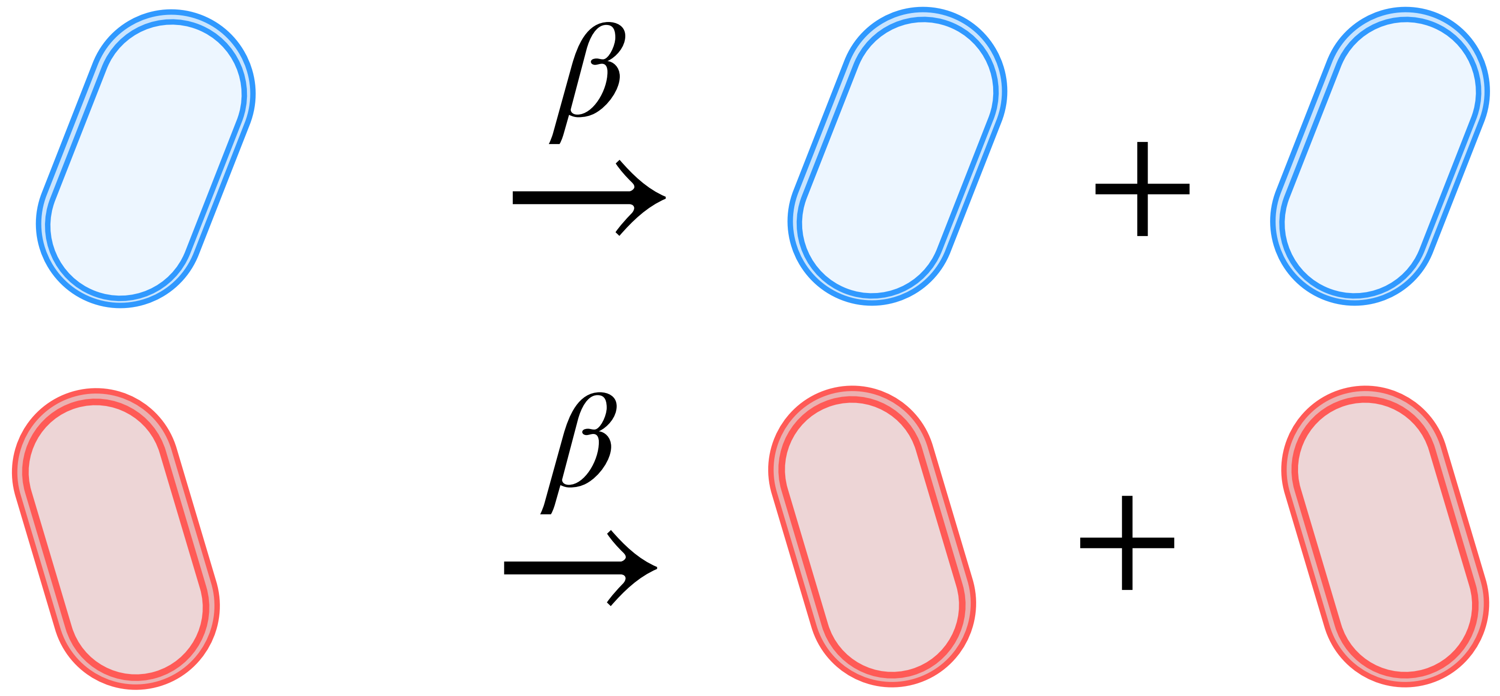

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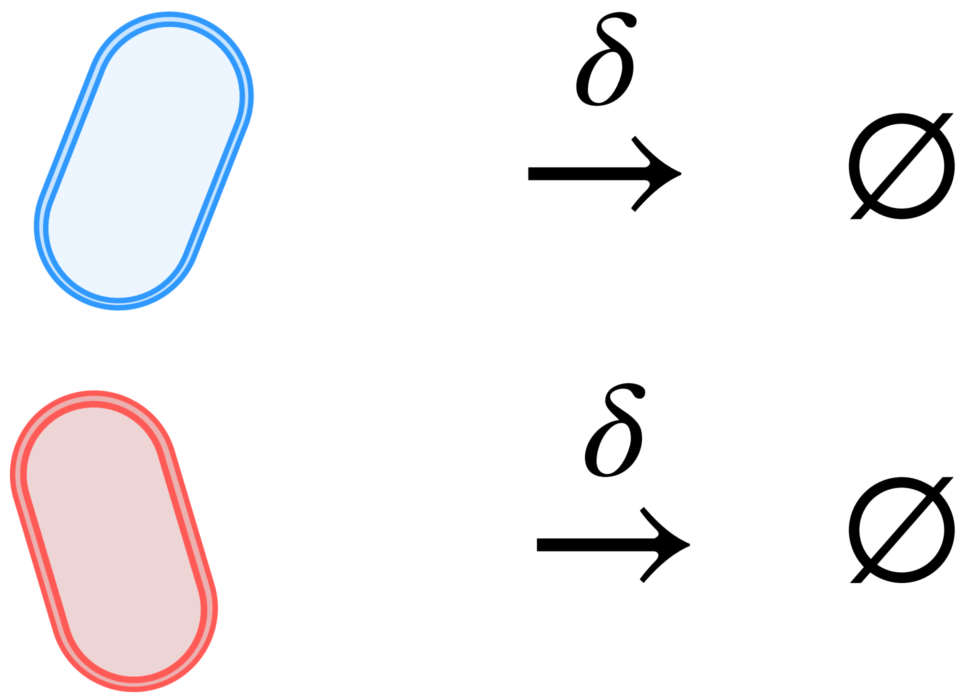
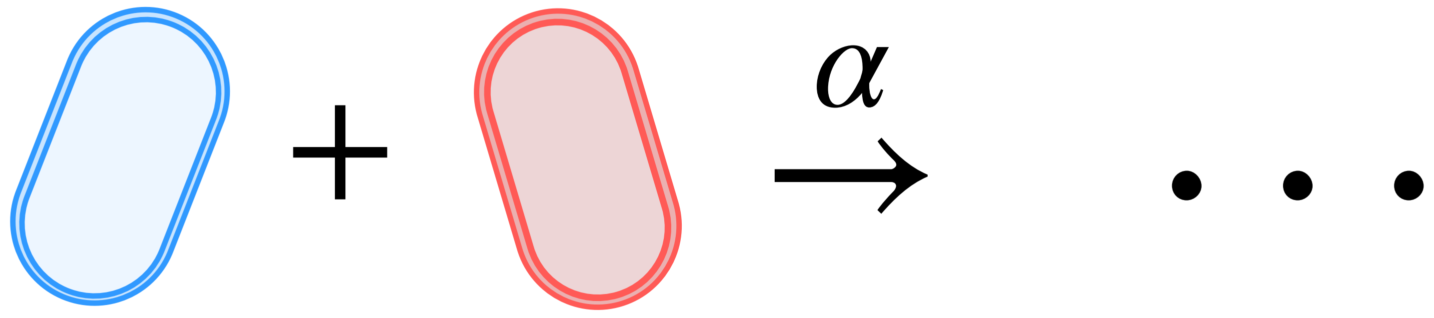
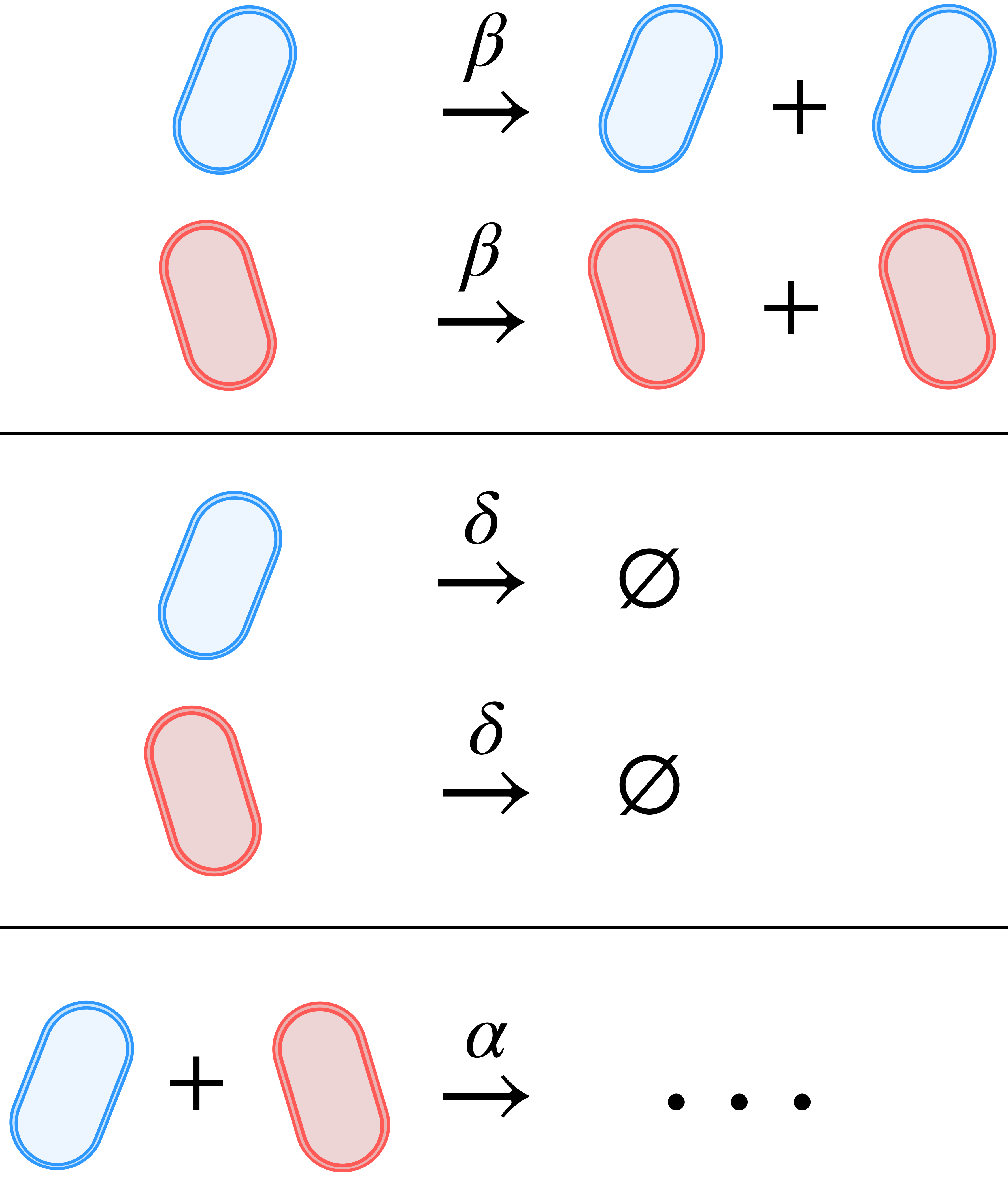


Interspecific competition

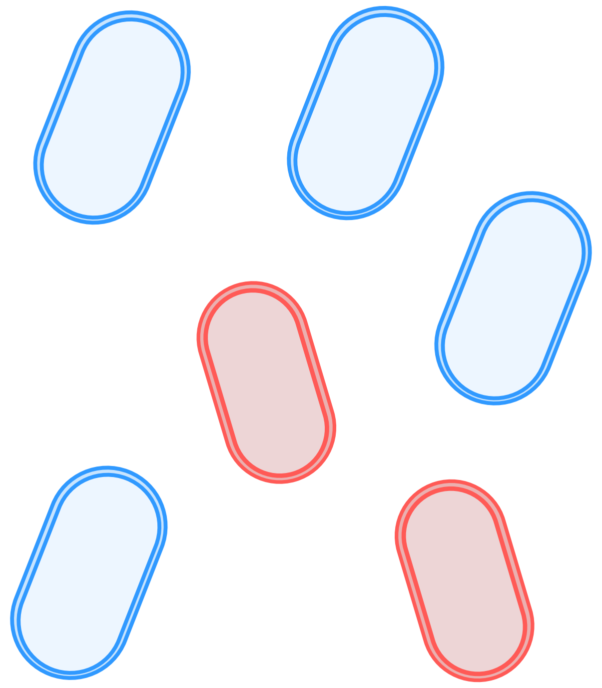


Competitive LV dynamics

		Propensity in state (A_t, B_t)
Reproduction		$\beta \cdot A_t$
		$\beta \cdot B_t$
Mortality		$\delta \cdot A_t$
		$\delta \cdot B_t$
Interspecific competition		$\alpha A_t B_t$

Microbial majority consensus

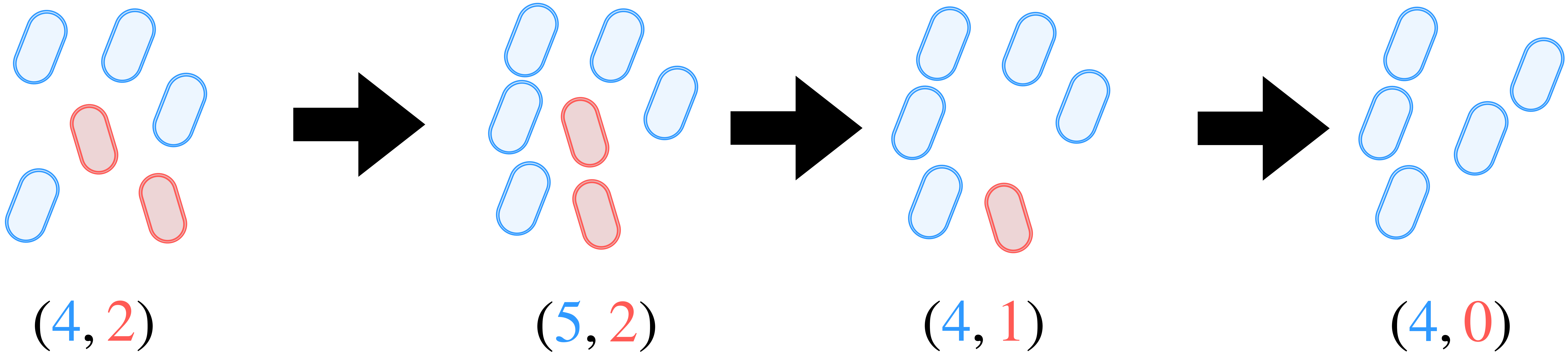
- Initial configuration $(A_0, B_0) \in \mathbb{N}^2$
  - initial gap $\Delta = |A_0 - B_0|$
  - initial population size $n = A_0 + B_0$



$(4, 2)$

Microbial majority consensus

- Initial configuration $(A_0, B_0) \in \mathbb{N}^2$
  - initial gap $\Delta = |A_0 - B_0|$
  - initial population size $n = A_0 + B_0$
- **Execution:** Markov chain $(A_t, B_t)_{t \geq 0}$



Microbial majority consensus

- Initial configuration $(A_0, B_0) \in \mathbb{N}^2$
  - initial gap $\Delta = |A_0 - B_0|$
  - initial population size $n = A_0 + B_0$

Question: How large does Δ need to be to reach *majority* consensus with high probability?

Competitive LV models:
interference competition

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interference competition

Self-destructive
("symmetric")

$$A + B \xrightarrow{\alpha} \varnothing$$

Competitive LV models:
interference competition

Self-destructive
(“symmetric”)

$$A + B \xrightarrow{\alpha} \varnothing$$

OR

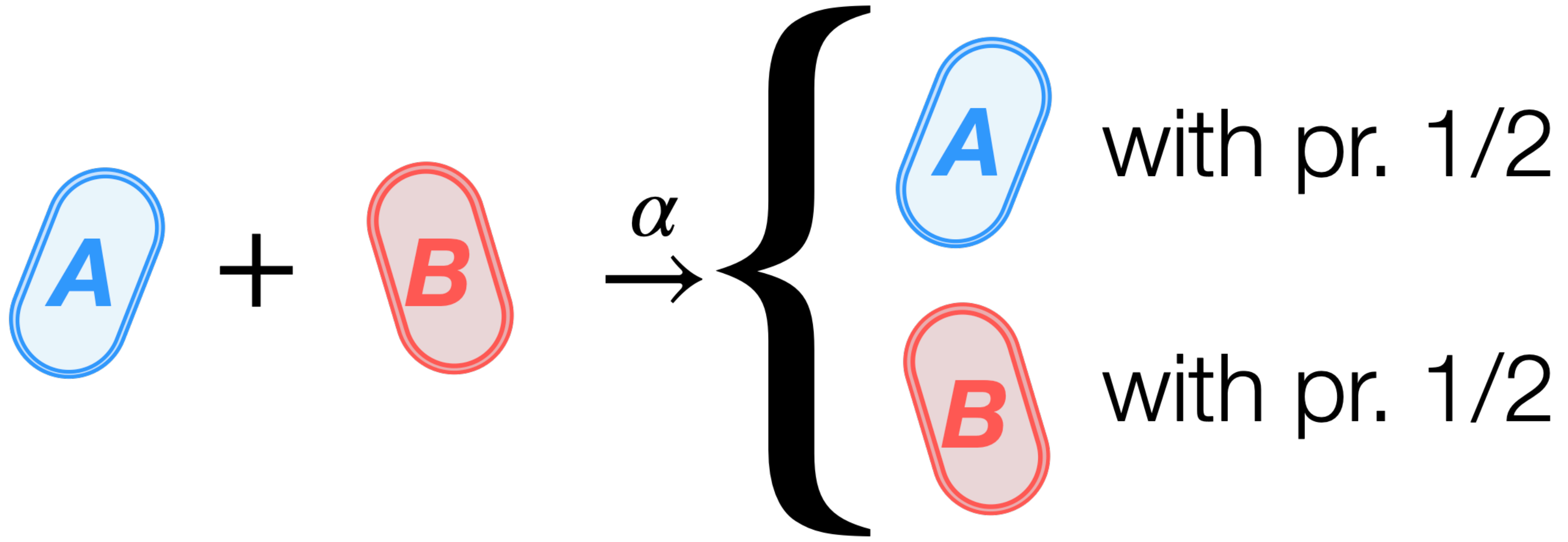
Non-self-destructive
(“asymmetric”)

$$A + B \xrightarrow{\alpha} \begin{cases} A & \text{with pr. } 1/2 \\ B & \text{with pr. } 1/2 \end{cases}$$

Prior work: cell mortality only via competition

Self-destructive

Non-self-destructive



Prior work: cell mortality only via competition

Self-destructive

$$A + B \xrightarrow{\alpha} \varnothing$$

- $O\left(\sqrt{n \log n}\right)$ gap sufficient w.h.p.
- no individual cell mortality ($\delta = 0$)

Cho, Függer, Hopper, Kushwaha, Nowak, Soubeyran (DISC 2019)

Non-self-destructive

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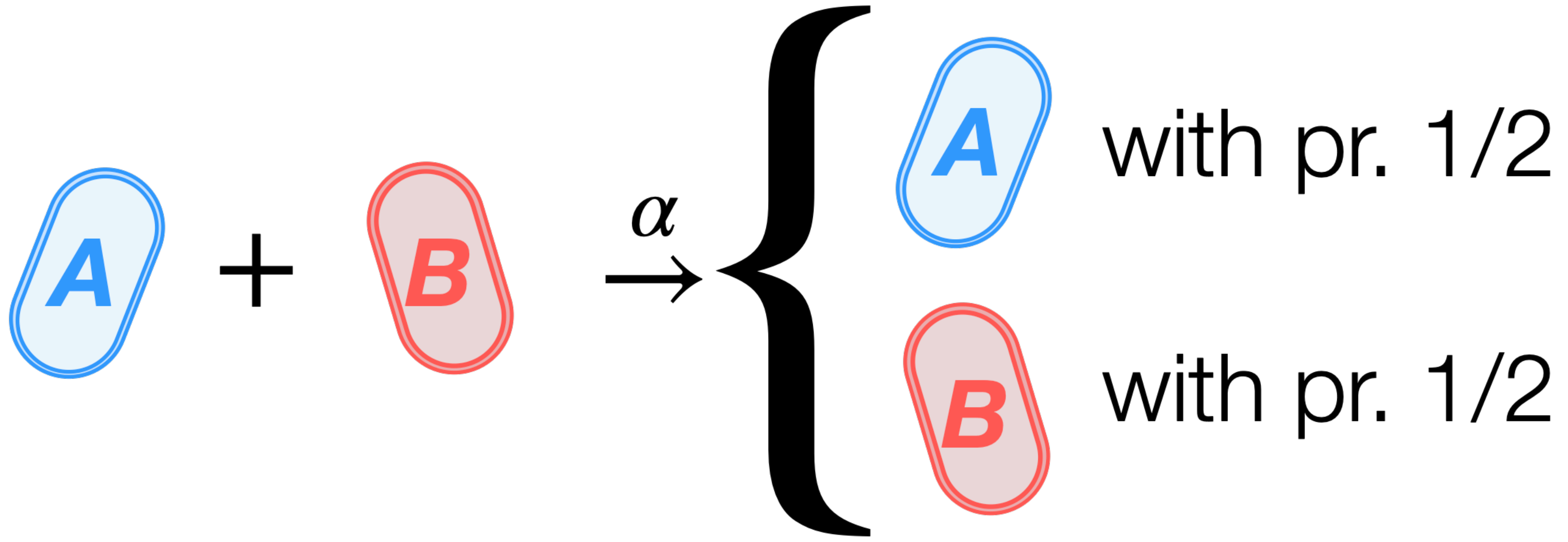

- $O\left(\sqrt{n \log n}\right)$ gap sufficient "w.h.p."
- birth via certain nutrient dynamics
- no individual cell mortality ($\delta = 0$)

Andaur, Burman, Függer, Kushwaha, Manssouri, Nowak, Rybicki (2021)

Recent results

Self-destructive

$$A + B \xrightarrow{\alpha} \varnothing$$

Függer, Nowak, Rybicki (PODC 2024)

Non-self-destructive

$$A + B \xrightarrow{\alpha} \begin{cases} A & \text{with pr. } 1/2 \\ B & \text{with pr. } 1/2 \end{cases}$$

Függer, Nowak, Rybicki (PODC 2024)

Recent results

Self-destructive

$$A + B \xrightarrow{\alpha} \varnothing$$

polylogarithmic gap Δ

necessary and sufficient!

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Recent results

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Függer, Nowak, Rybicki (PODC 2024)

Non-self-destructive

$$A + B \xrightarrow{\alpha} \begin{cases} A & \text{with pr. } 1/2 \\ B & \text{with pr. } 1/2 \end{cases}$$

polynomial gap Δ
necessary and sufficient!

Függer, Nowak, Rybicki (PODC 2024)

Recent results

Self-destructive

$$A + B \xrightarrow{\alpha} \emptyset$$

$$\Omega\left(\sqrt{\log n}\right) - O\left(\log^2 n\right)$$

Függer, Nowak, Rybicki (PODC 2024)

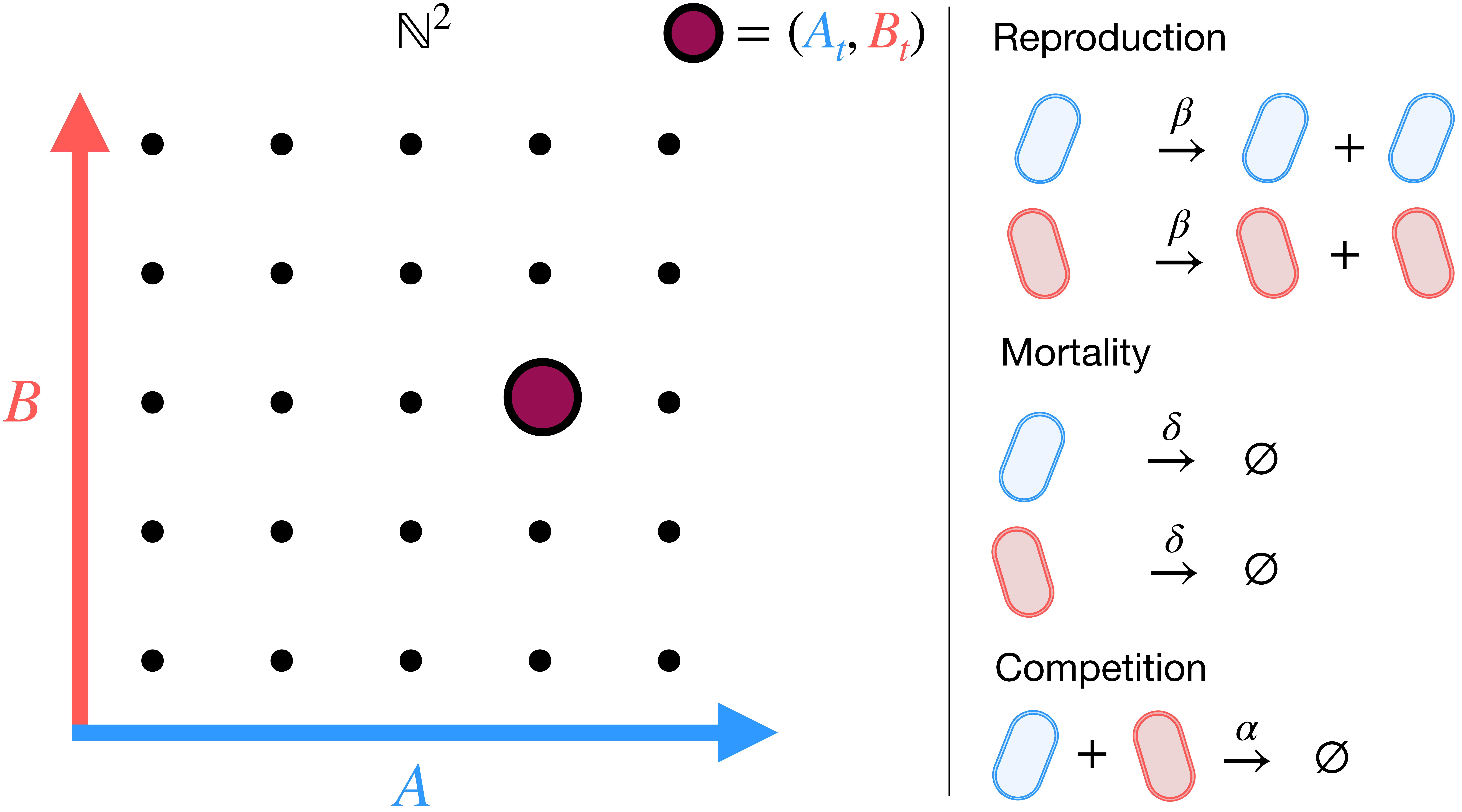
Non-self-destructive

$$A + B \xrightarrow{\alpha} \begin{cases} A & \text{with pr. } 1/2 \\ B & \text{with pr. } 1/2 \end{cases}$$

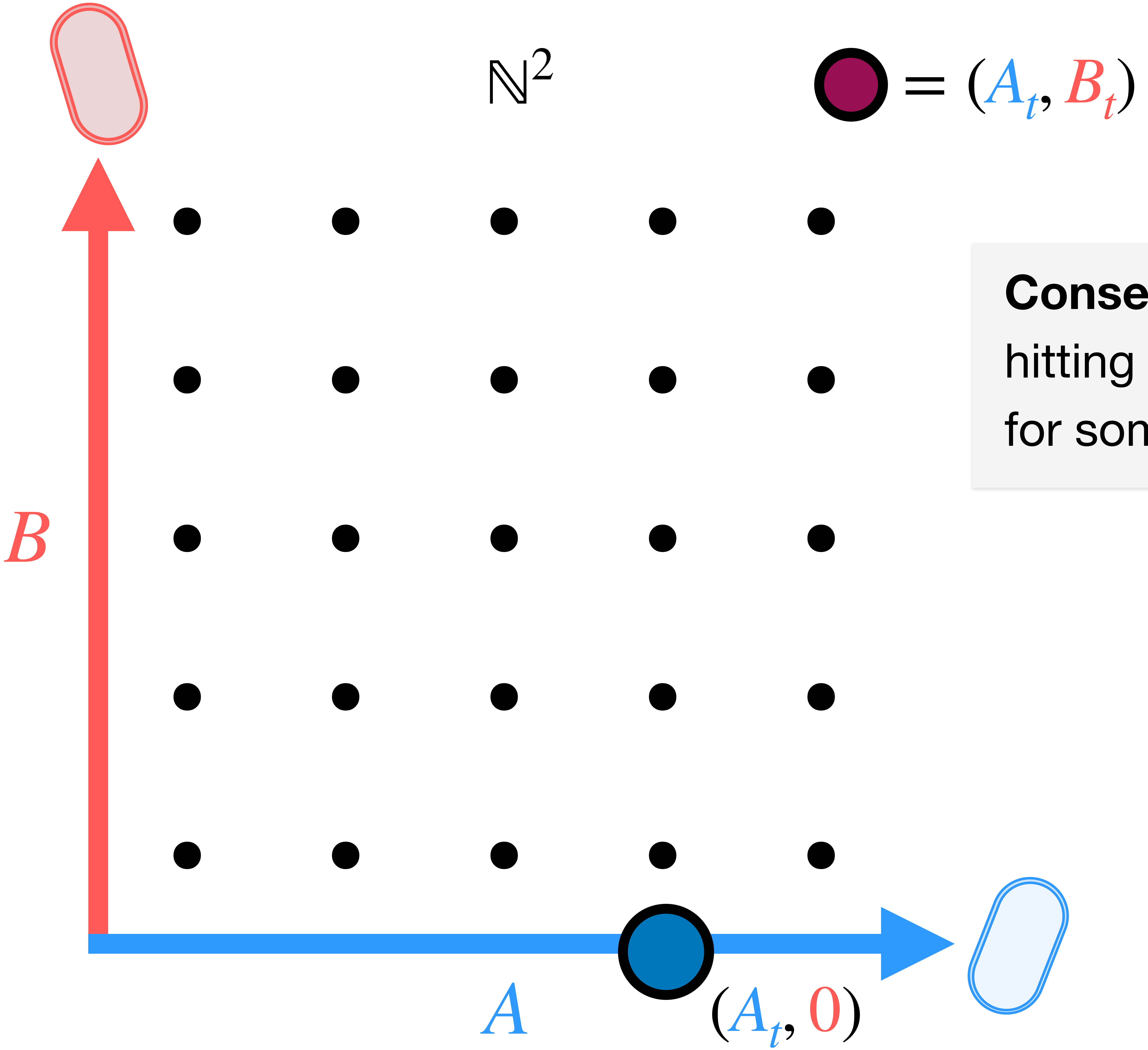
$$\Omega\left(\sqrt{n}\right) - O\left(\sqrt{n \log n}\right)$$

Függer, Nowak, Rybicki (PODC 2024)

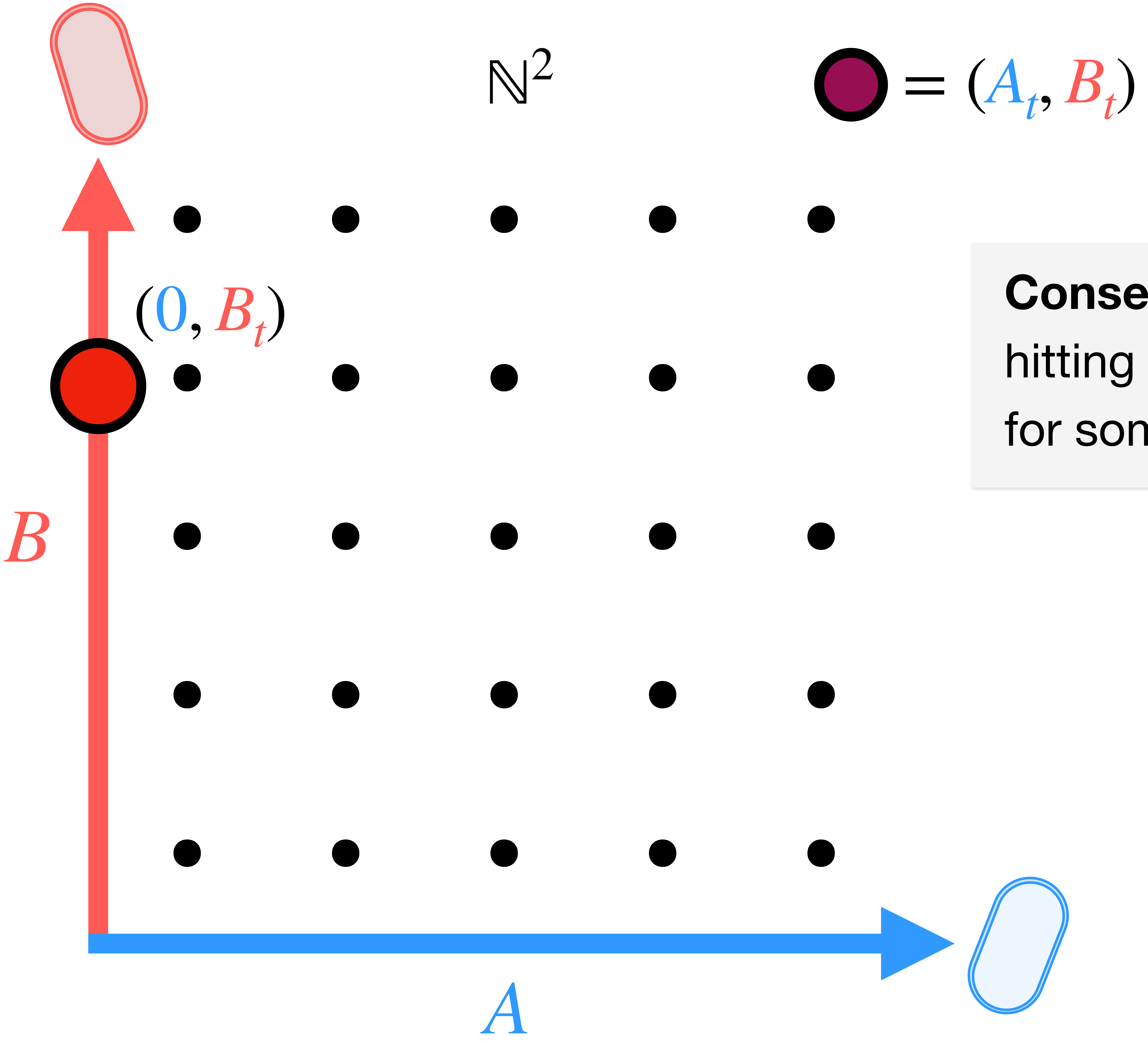
The dominating chain technique



$\mathbb{N}^2$
$= (A_t, B_t)$
B
A
Reproduction
$\xrightarrow{\beta}$ +
$\xrightarrow{\beta}$ +
Mortality
$\xrightarrow{\delta}$ $\varnothing$
$\xrightarrow{\delta}$ $\varnothing$
Competition
+ $\xrightarrow{\alpha}$ $\varnothing$



Consensus time T =
hitting time to $(a, 0)$ or $(0, b)$
for some $a, b > 0$



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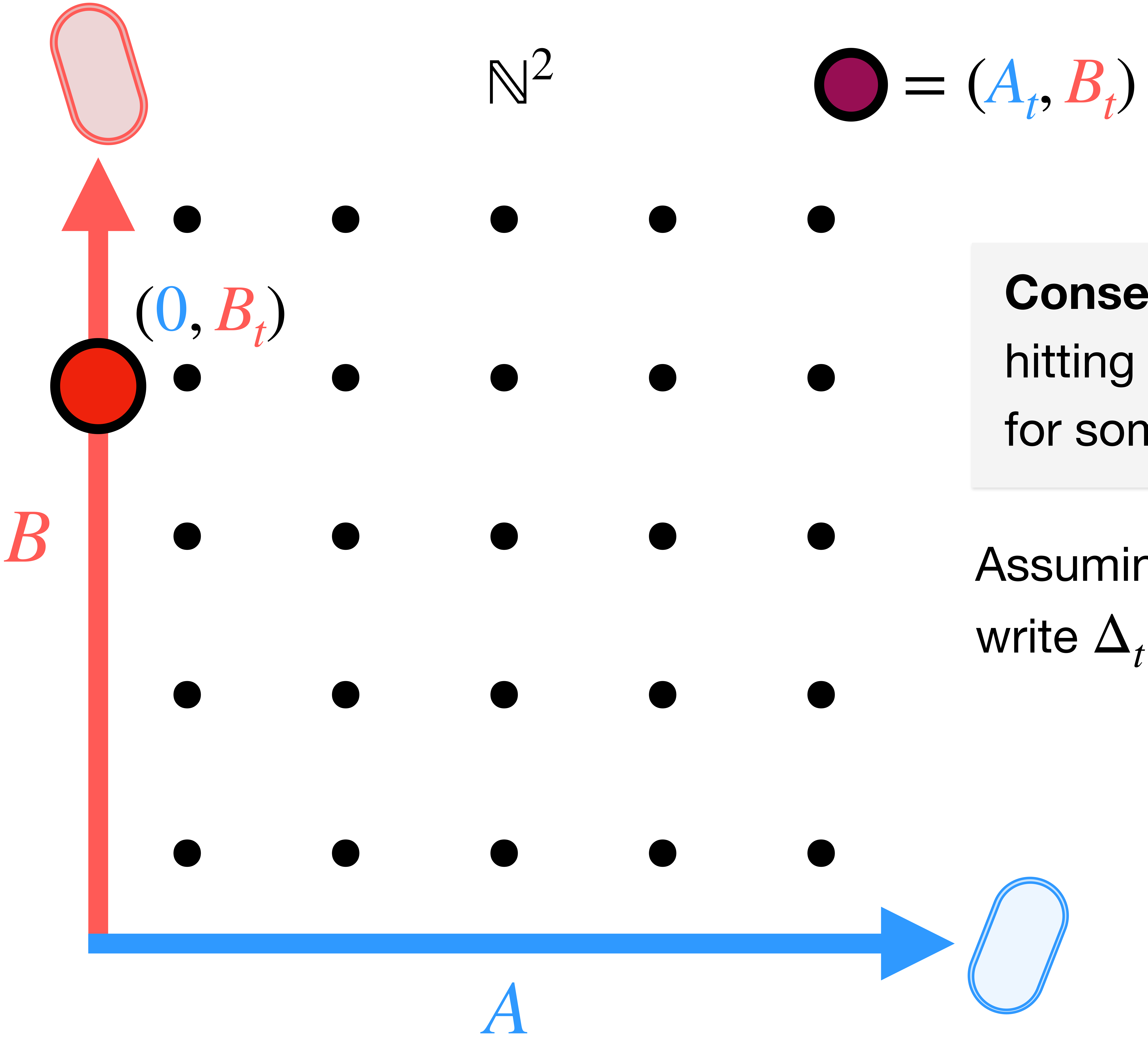

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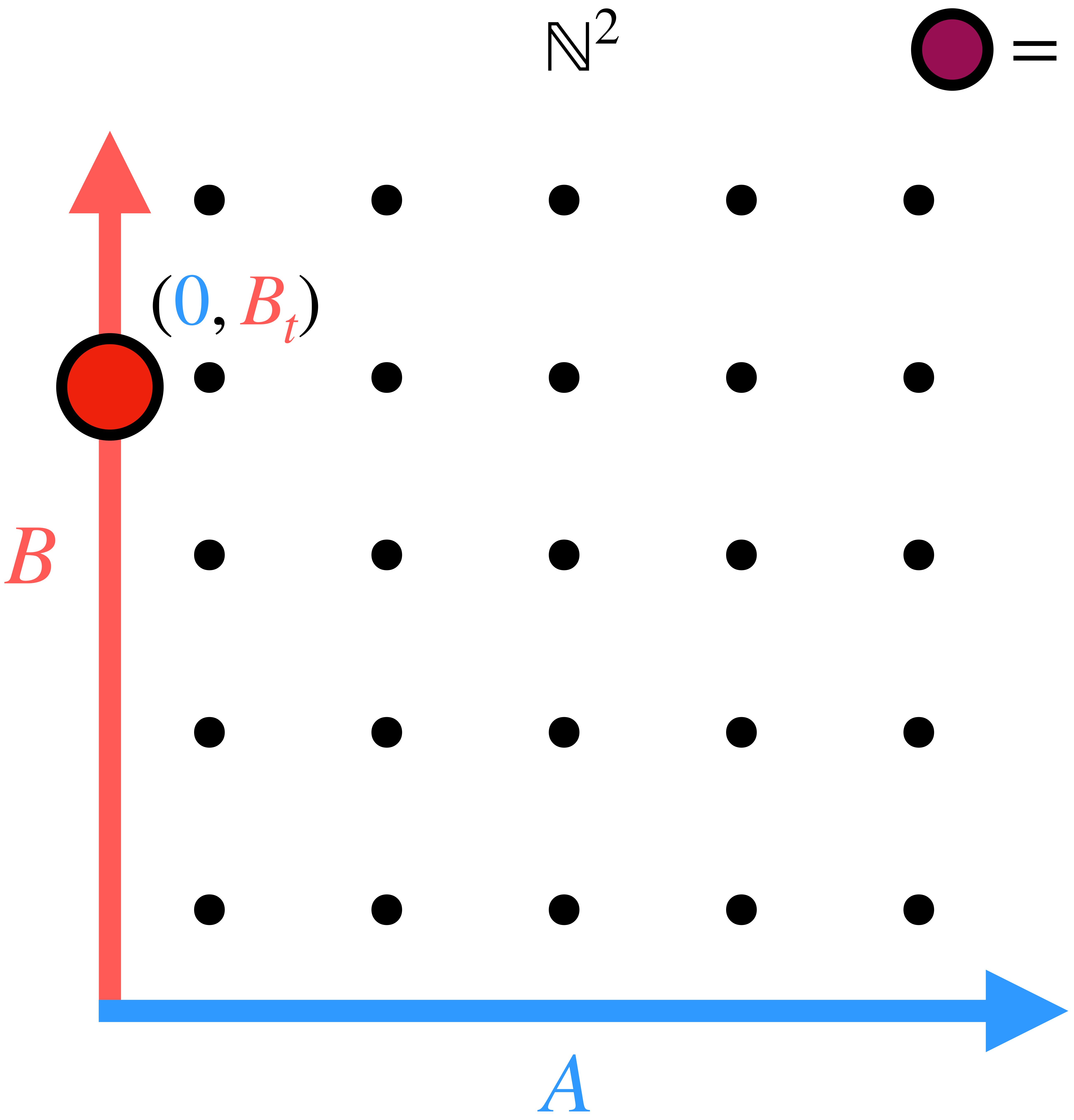
Assuming $A_0 > B_0$, write $\Delta_t = A_t - B_t$



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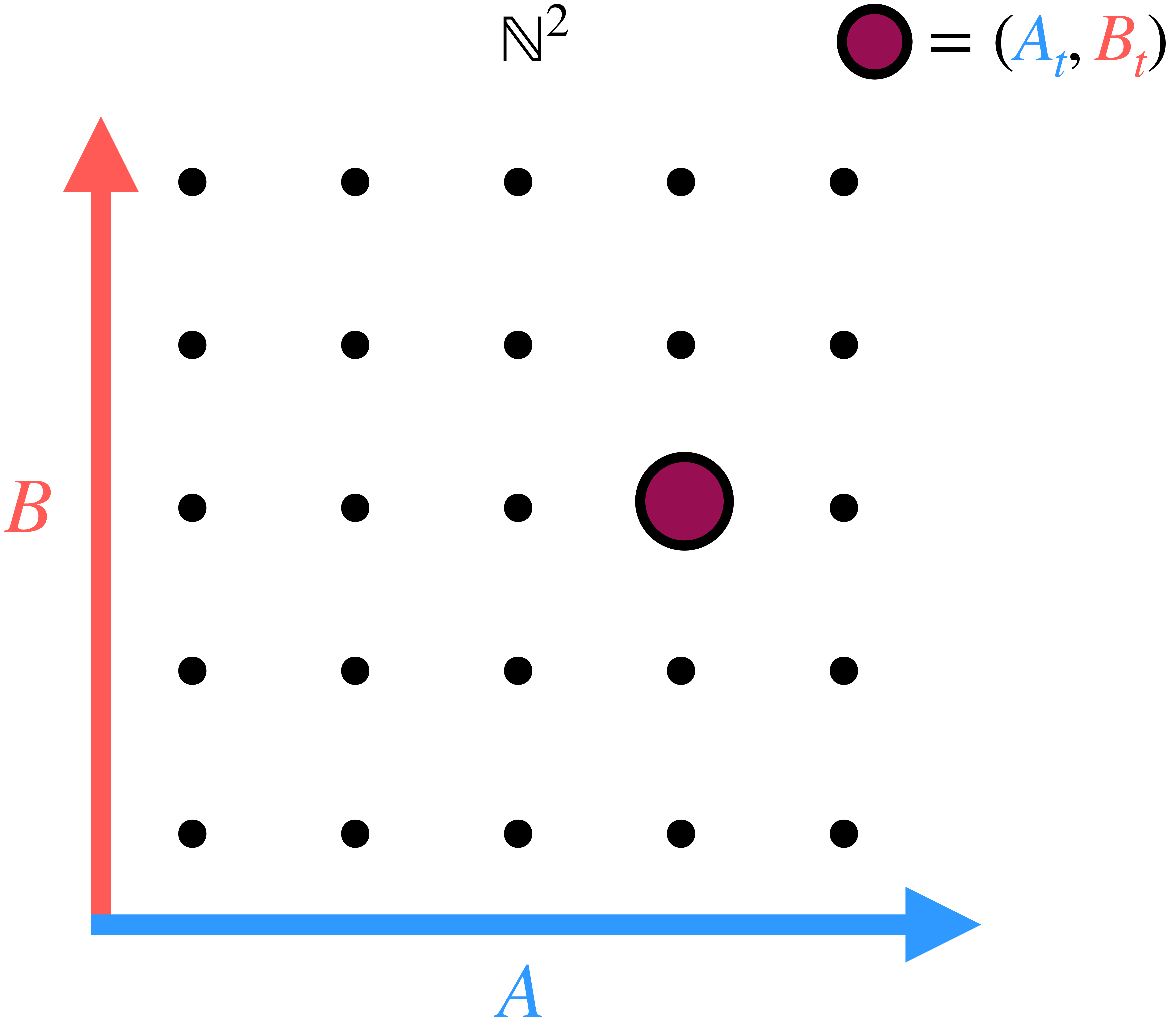
Assuming $A_0 > B_0$,
write $\Delta_t = A_t - B_t$

Probability of majority consensus =
$\Pr[\Delta_T > 0]$



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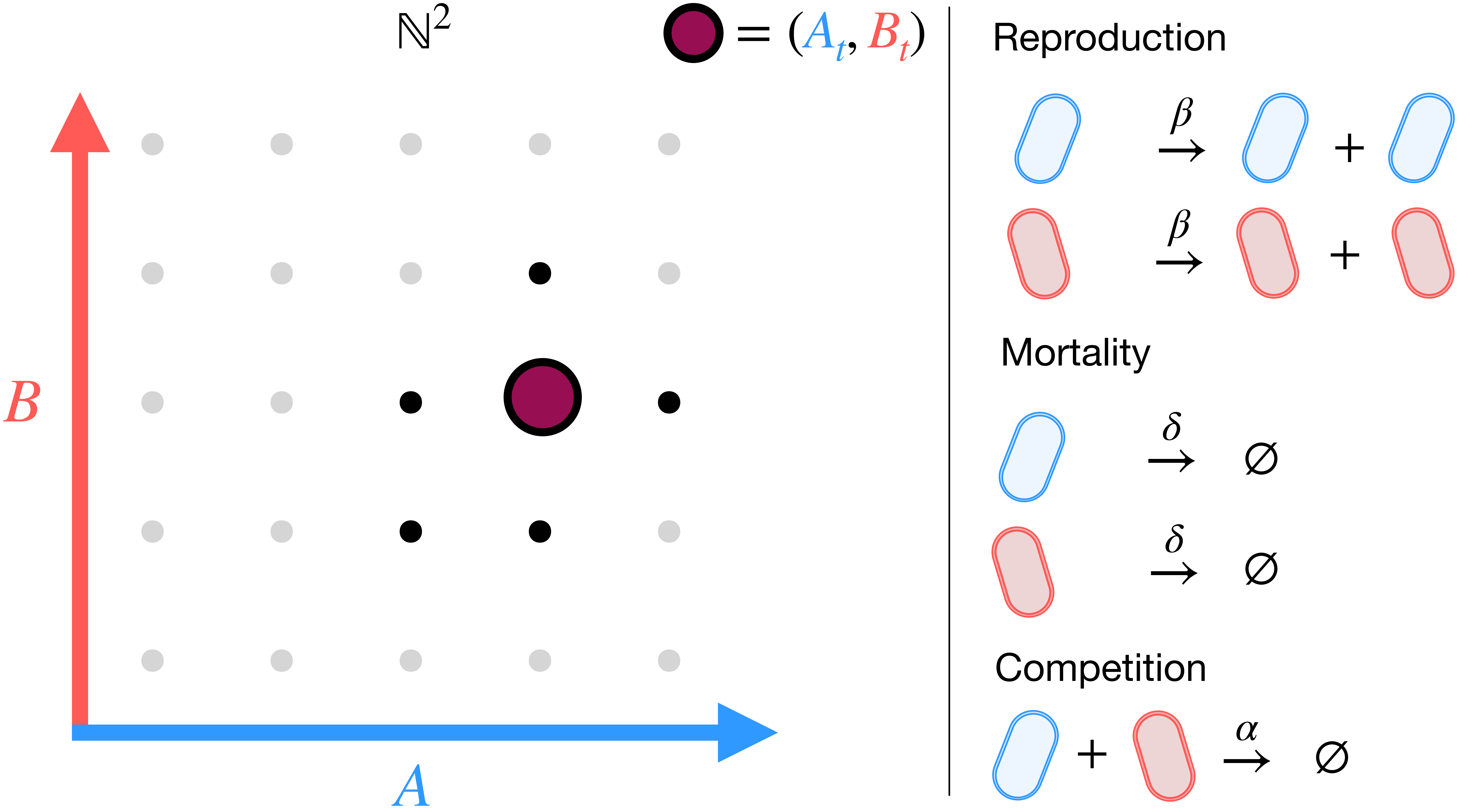

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βA_t
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A
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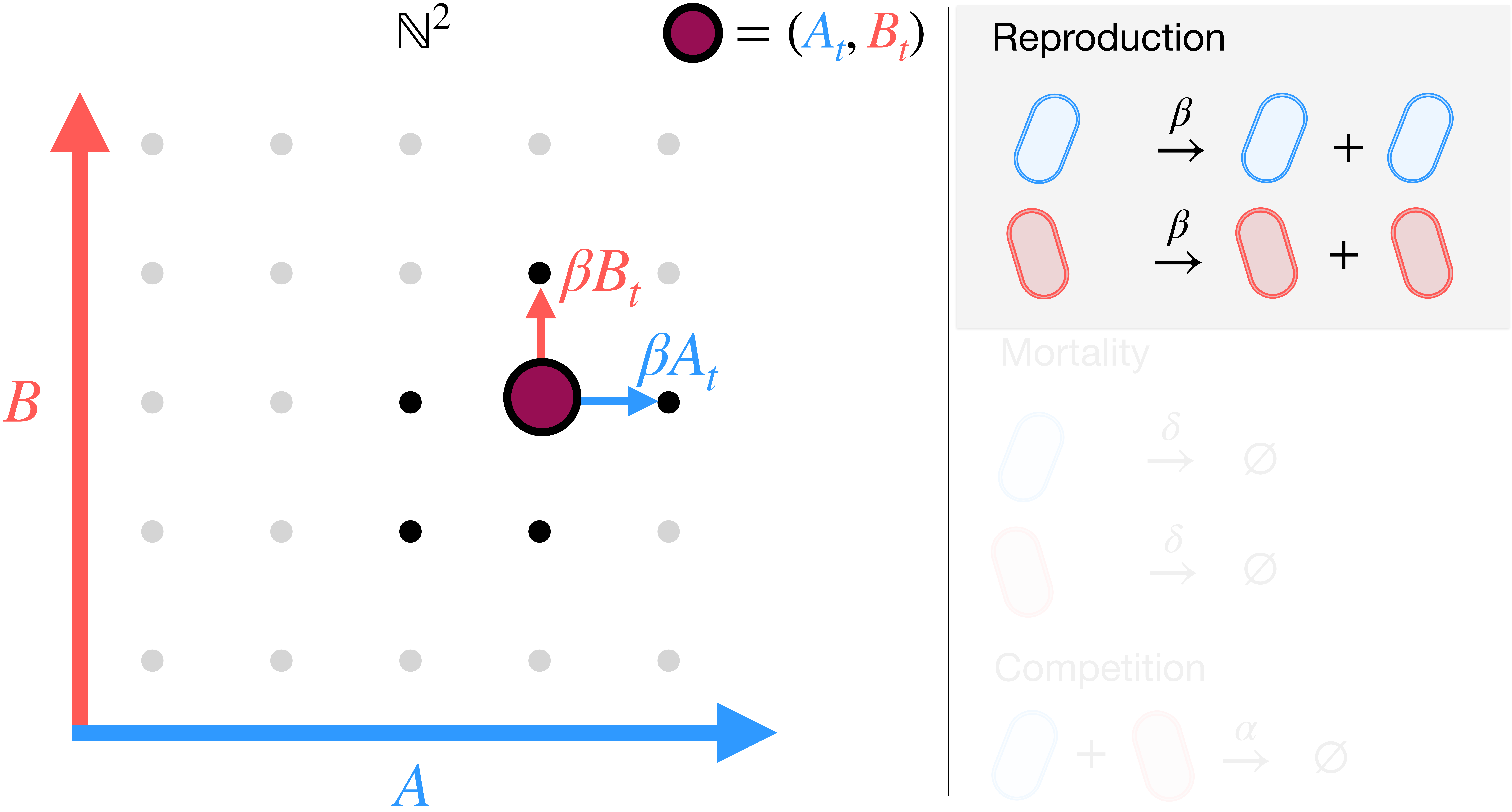
$\mathbb{N}^2$
$= (A_t, B_t)$
B
A
βA_t
Reproduction
β
+
β
+



$\mathbb{N}^2$
$= (A_t, B_t)$
βB_t
βA_t
B
A
Reproduction
β
β
Mortality
δ
δ
Competition
α

$\mathbb{N}^2$
$= (A_t, B_t)$
δA_t
δB_t
B
A
Reproduction
β
$+$
β
$+$
Mortality
δ
$\varnothing$
δ
$\varnothing$
Competition
$+$
α
$\varnothing$

$\mathbb{N}^2$
$= (A_t, B_t)$
B
A
$\alpha A_t B_t$
Reproduction
β
$+$
β
$+$
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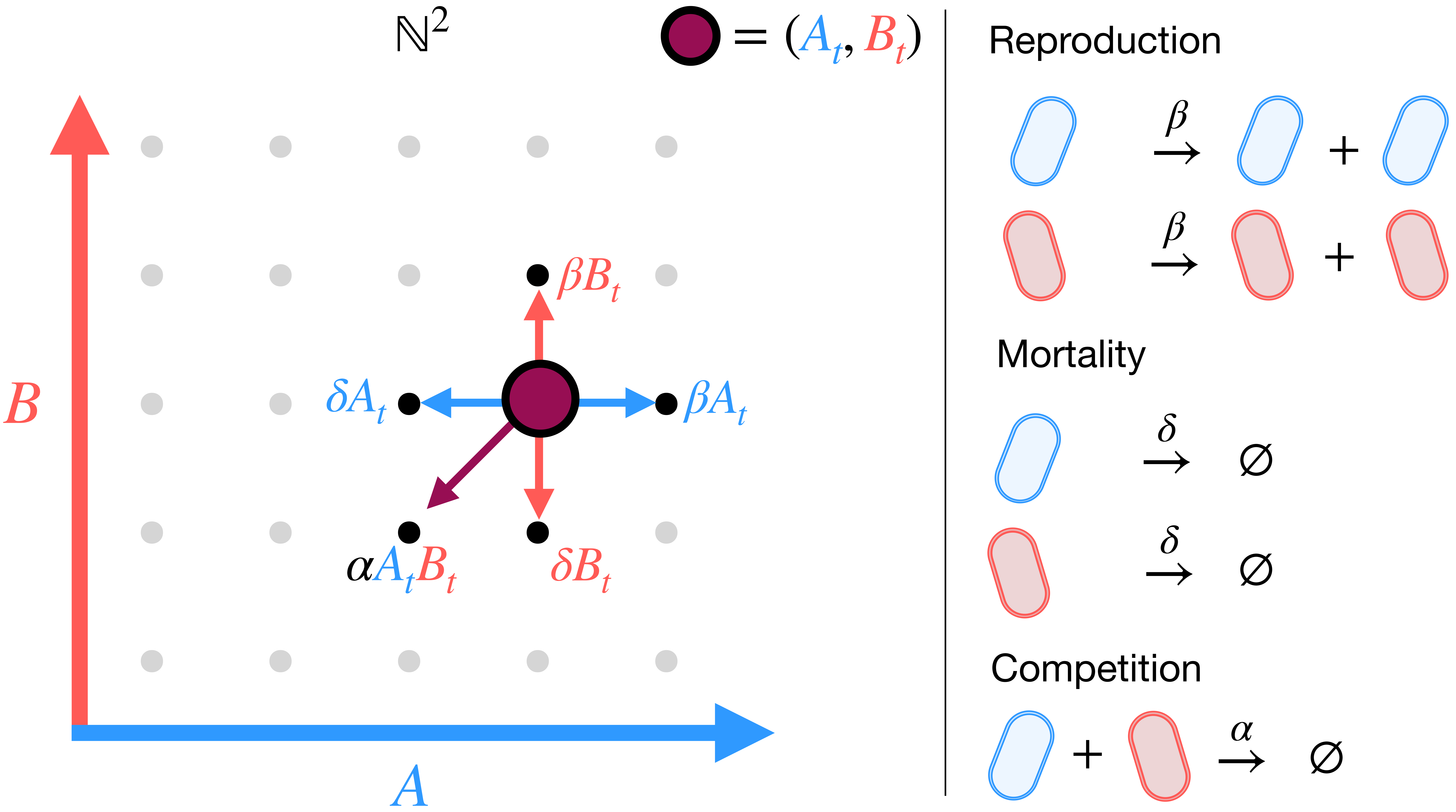
$\varnothing$
Competition
$+$
α
$\varnothing$



$\mathbb{N}^2$
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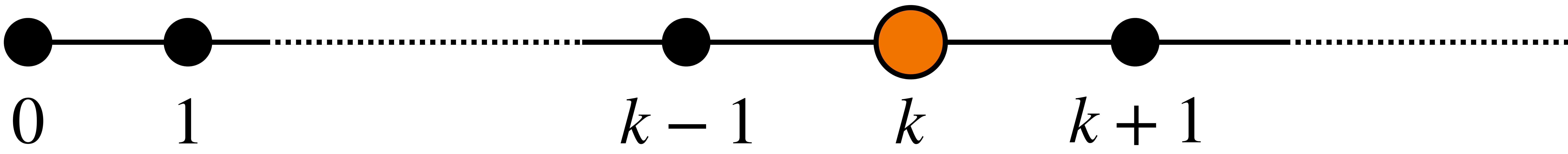
Dominating chain technique

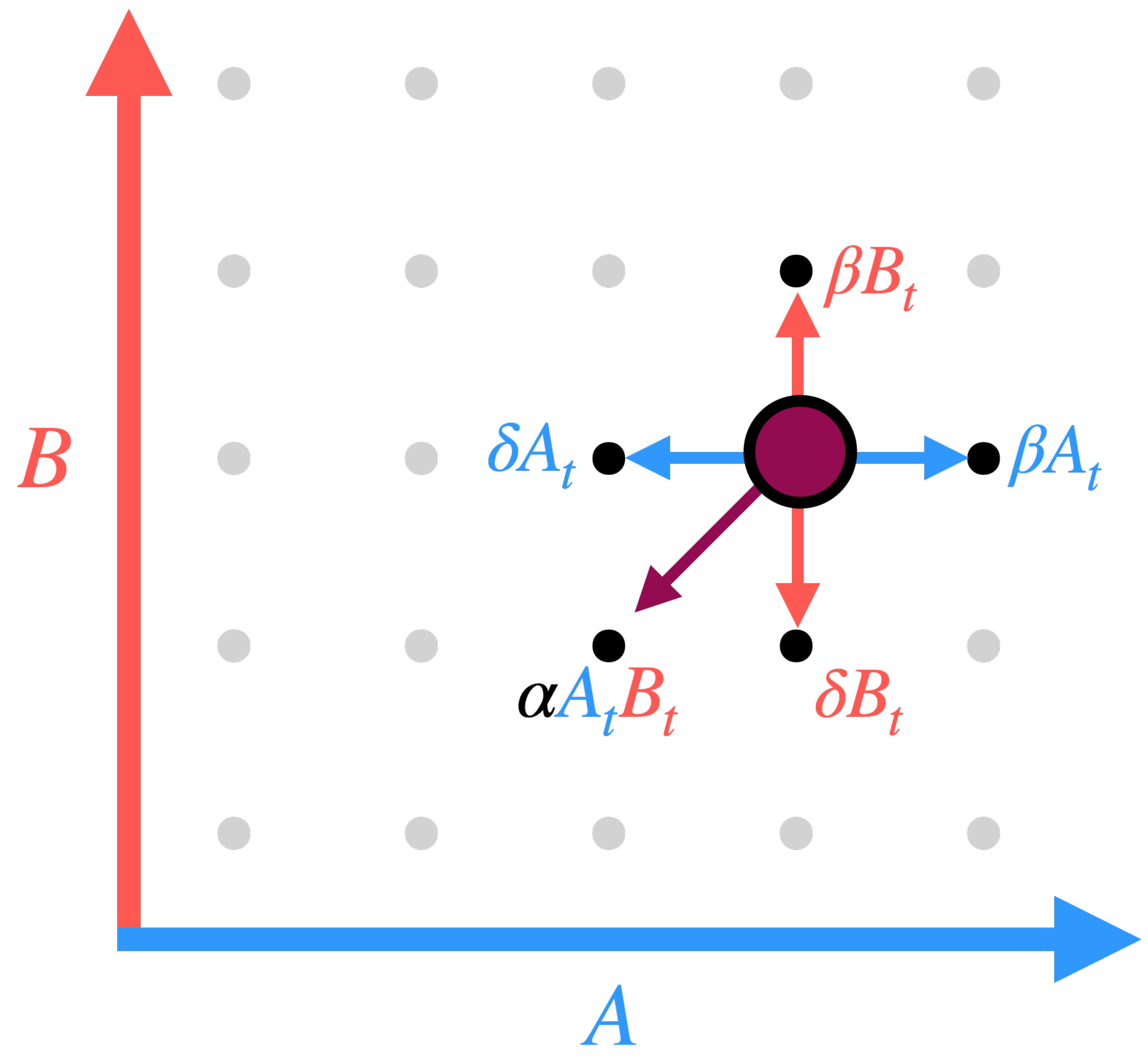
Stochastic domination

Find a *single-species* birth-death chain $(N_t)_{t \geq 0}$

$(N_t)_{t \geq 0}$



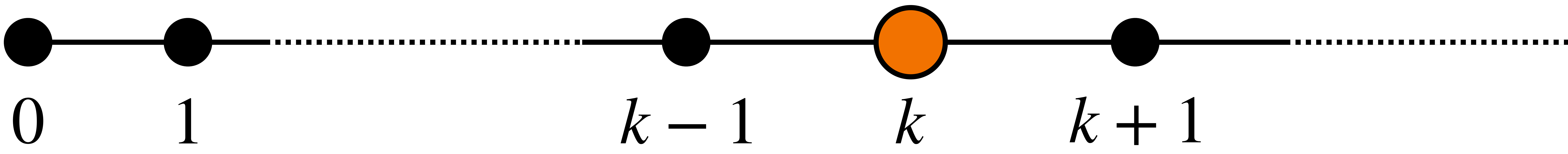
Dominating chain technique



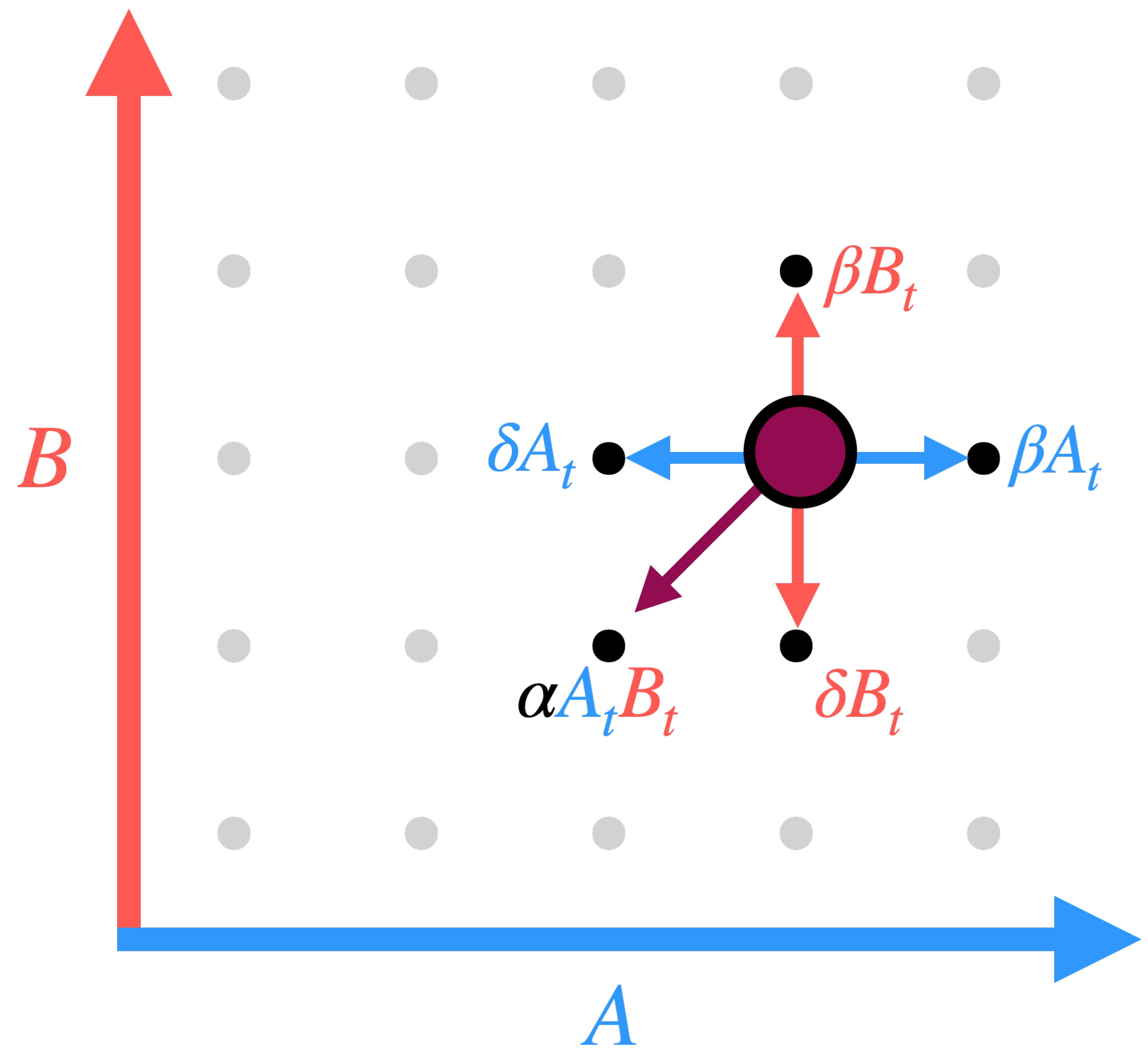
Stochastic domination

Find a *single-species* birth-death chain $(N_t)_{t \geq 0}$ that stochastically dominates $\min(A_t, B_t) \preceq N_t$

$(N_t)_{t \geq 0}$



Dominating chain technique



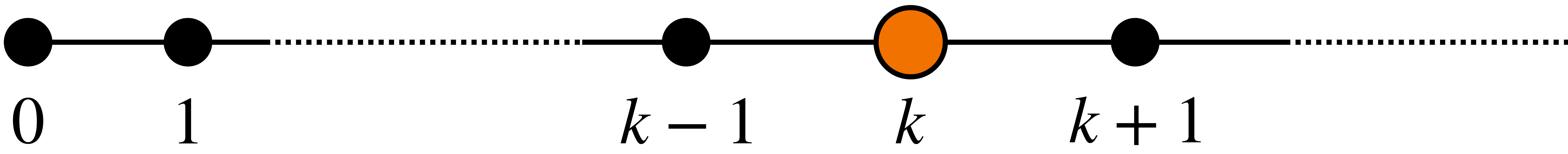
Stochastic domination

Find a *single-species* birth-death chain $(N_t)_{t \geq 0}$ that stochastically dominates $\min(A_t, B_t) \preceq N_t$

In state k,

- birth probability $p(k) = O(1/k)$
- death probability $q(k) = \Omega(1)$

$(N_t)_{t \geq 0}$



Dominating chain technique:
self-destructive competition

LV chain		**Single-species chain**
consensus time of $(A_t, B_t)_{t\geq 0}$	$\preceq$	absorption time of $(N_t)_{t\geq 0}$

Dominating chain technique:
self-destructive competition

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w.h.p. $O(n)$		w.h.p. $O(n)$

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$$\Delta_t = A_t - B_t$$

Dominating chain technique:
self-destructive competition

LV chain		**Single-species chain**
consensus time of $(A_t, B_t)_{t\geq 0}$	$\preceq$	absorption time of $(N_t)_{t\geq 0}$
w.h.p. $O(n)$		w.h.p. $O(n)$
# steps that decrease $\Delta_t = A_t - B_t$ before consensus time	$\preceq$	# steps that increase N_t

Dominating chain technique:
self-destructive competition

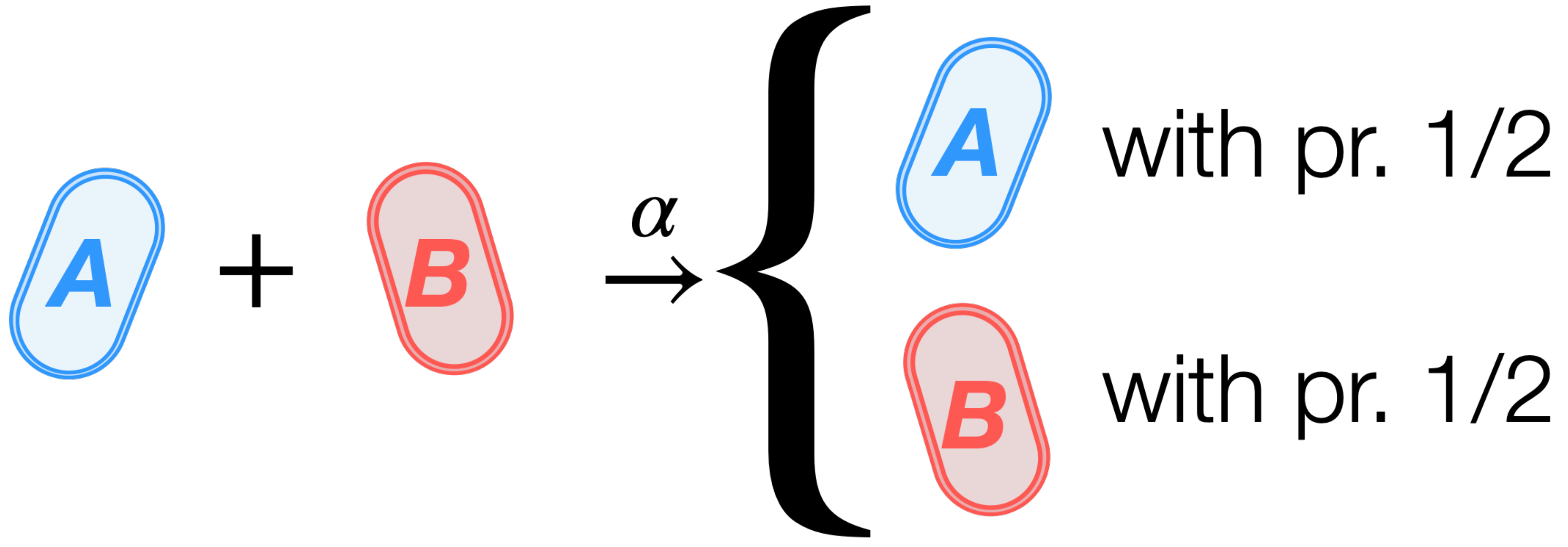
LV chain		**Single-species chain**
consensus time of $(A_t, B_t)_{t \geq 0}$	$\preceq$	absorption time of $(N_t)_{t \geq 0}$
w.h.p. $O(n)$		w.h.p. $O(n)$
# steps that decrease $\Delta_t = A_t - B_t$ before consensus time	$\preceq$	# steps that increase N_t
w.h.p. $O(\log^2 n)$		w.h.p. $O(\log^2 n)$

Recent results

Self-destructive

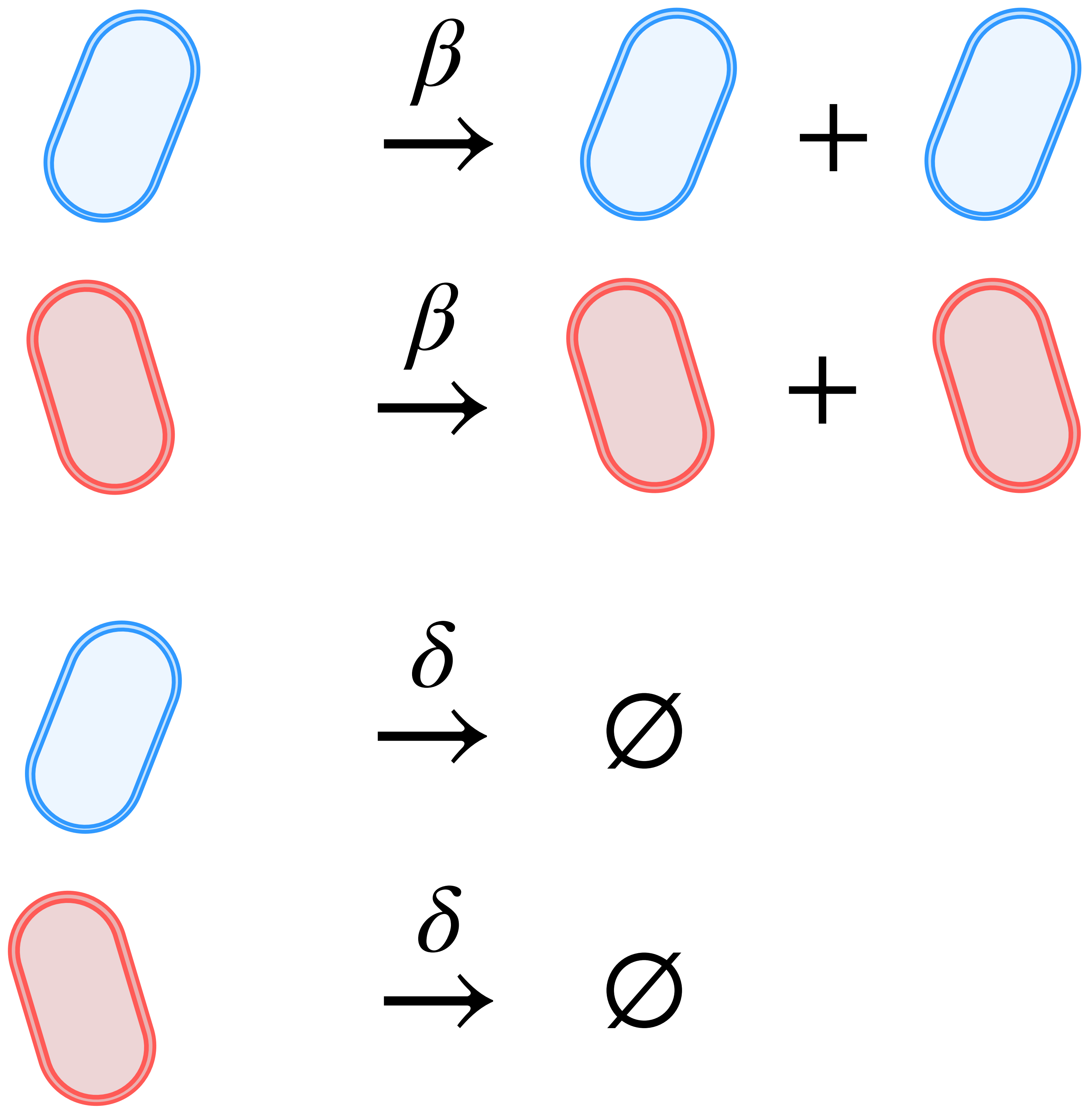
$$\Delta_0 = O\left(\log^2 n\right) \text{ suffices}$$

Non-self-destructive



$$\Delta_0 = O\left(\sqrt{n \log n}\right) \text{ suffices}$$

What about **no** competition?



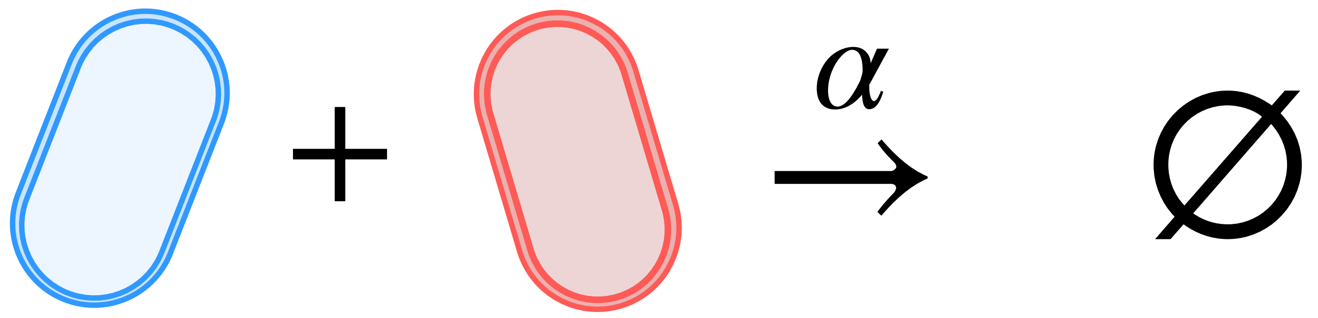
- independent birth-death processes
- probability that ***A*** “wins” is

$$\frac{A_0}{A_0 + B_0}$$

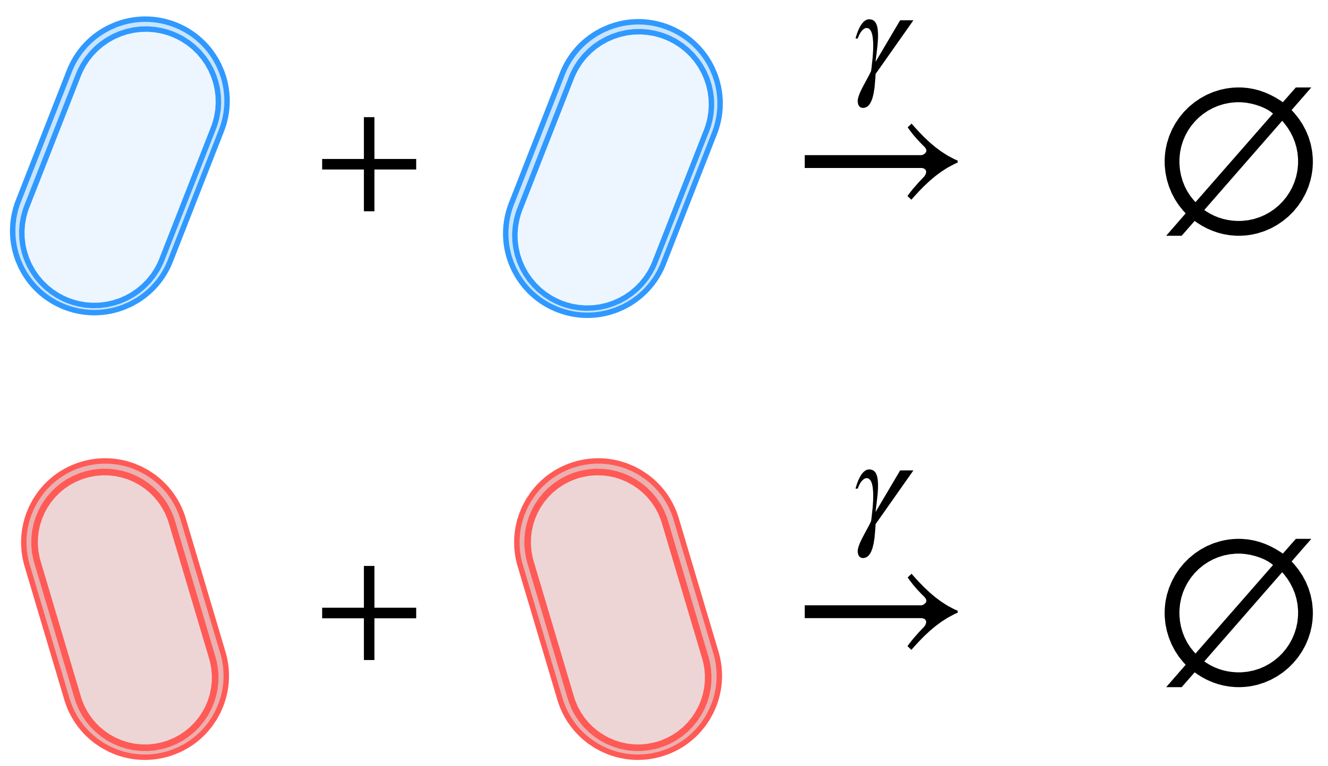
“Folklore” /
Andaur, Burman, Függer, Kushwaha, Manssouri, Nowak, Rybicki (2021)

What about **intraspecific** competition?

Interspecific competition



Intraspecific competition



- if $\gamma \approx \alpha$, then probability that ***A*** "wins" is

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Függer, Nowak, Rybicki (PODC 2024)

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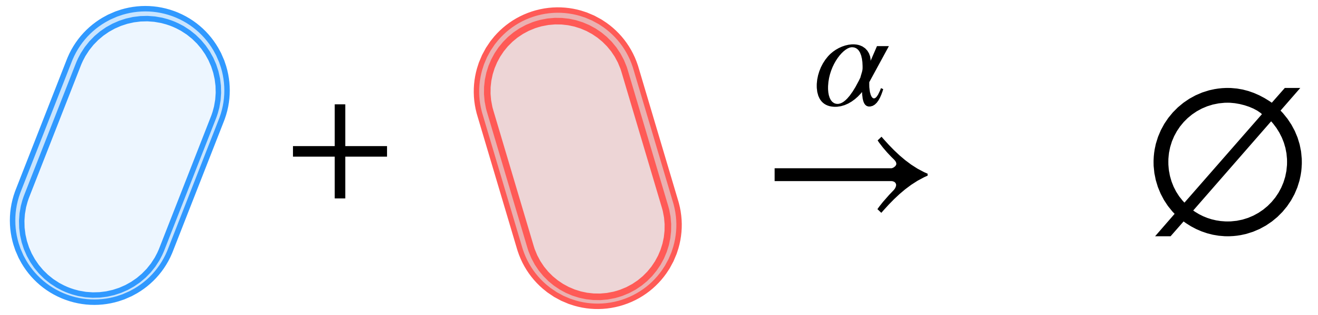
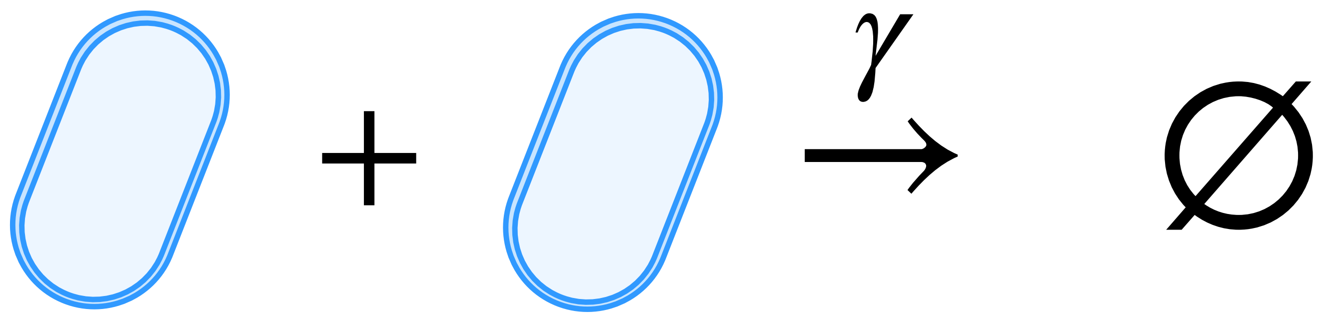
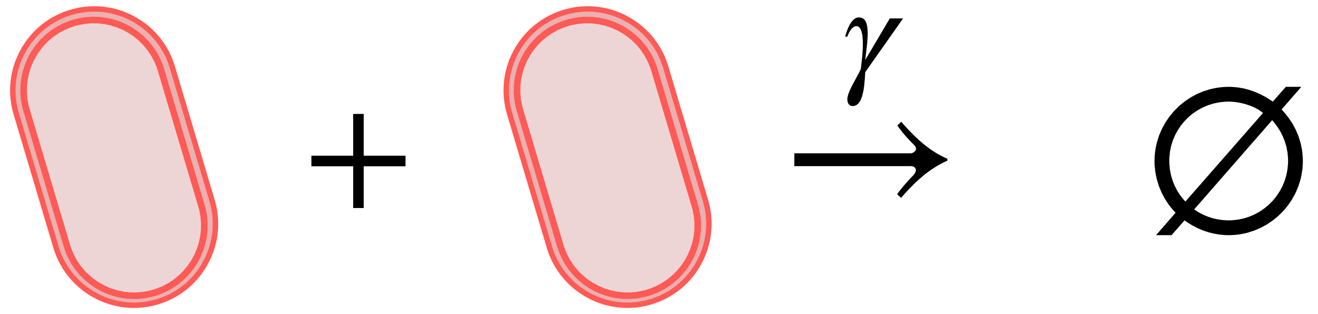
Interspecific competition



Intraspecific competition





- if $\gamma \approx \alpha$, then probability that ***A*** "wins" is

$$\frac{A_0}{A_0 + B_0}$$

Open problem:

What happens with small $\gamma > 0$?

Today: $\gamma = 0$

Függer, Nowak, Rybicki (PODC 2024)

Closing thoughts

- can analyse (simple) individual-based models with **ecological processes**

- **sensitivity to noise** depends on mode of competition (and kinetics)

- **Open problems**
  — dealing with intraspecific competition?
  — beyond mass action kinetics?
  — resource-consumer dynamics?

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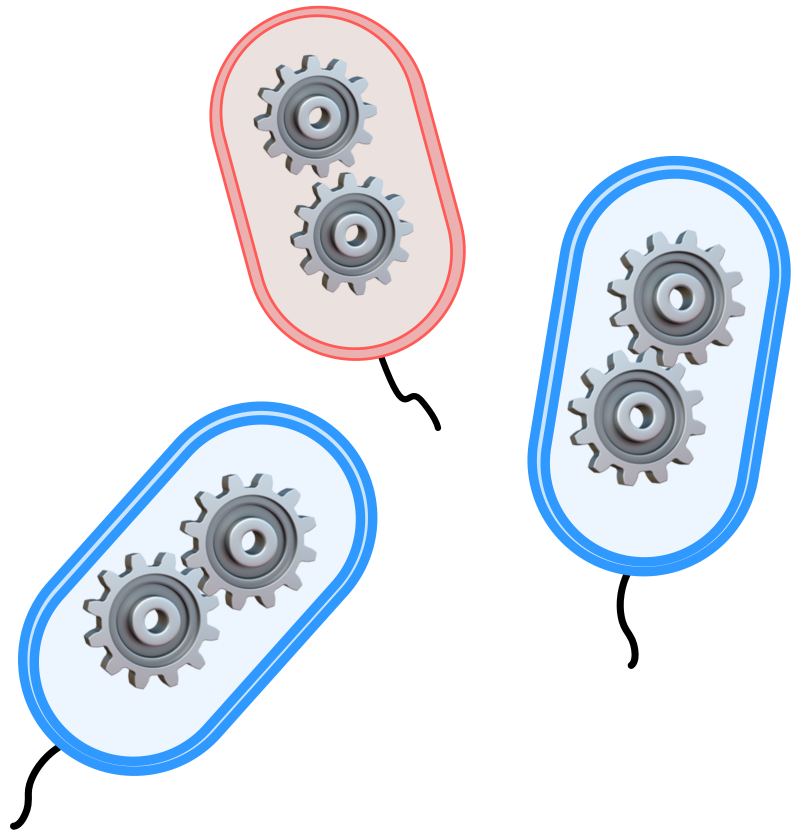

Thank you!